In this lecture we show that an **universal non-trivial minorant of** \( \text{M}(\alpha) \) for \( \alpha \) a reciprocal nonzero real \( > 1 \) algebraic integer which is not root of unity can be obtained from the class of integer polynomials

\[
\mathcal{C} := \{-1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} : \\
n \geq 3, \ m_1 - n \geq n - 1, \ m_q - m_{q-1} \geq n - 1 \ \text{for} \ 2 \leq q \leq s\}.
\]

More precisely : from the curve formed by the lenticular roots of the \( Ps \) of \( \mathcal{C} \).
Recall: for $P \in \mathcal{C}$, say $P(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s}$, with

$n \geq 3, m_1 - n \geq n - 1, m_q - m_{q-1} \geq n - 1$ for $2 \leq q \leq s$,

$$P(x) = A(x)B(x)C(x)$$

with

- $A(x)$ product of cyclotomic polynomials, often trivial (75 %),

- $B(x)$ product of reciprocal non-cyclotomic irreducible factors, conjectured to be inexistant,

- $C(x)$ irreducible non-reciprocal (unique factor), vanishing at the unique zero of $P$ in $(0, 1)$.

Let $\gamma$ be the unique zero of $P(x)$ in $(0, 1)$: $C(\gamma) = 0$.

The lenticulus of roots of $P$ is composed of Galois conjugates of $C$, conjugated with $\gamma$, and $\gamma$ is non-reciprocal. Existence domain: $\Re(z) > 1/2$. 
Example with $n = 37$:

$$P(x) := (-1 + x + x^{37}) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649}$$

**Figure**: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $P(x) = G_{37}(x) + \ldots + x^{649}$. The lenticulus of roots of $P$ is obtained by a very slight deformation of the restriction of the lenticulus of roots of $G_{37}$ to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of $P$ can be found in a narrow annular neighbourhood of $|z| = 1$. 
Q1: Is it possible to solve the Problem of Lehmer for the subcollection of \( \mathcal{C} \) of the trinomials \((-1 + x + x^n)_{n \geq 3}\) using the lenticuli of roots in \( \Re > 1/2 \)?

Yes + method asymptotic expansions of the roots. Limit Mahler measure = 1.38....

Q2: Is it possible to solve the Problem of Lehmer for the complete collection \( \mathcal{C} \) of polynomials \((P)_{n,m_1,m_2,...,m_s}\) using the curves formed by their lenticuli of roots in \( \Re > 1/2 \), or an angular part of them, without taking into account the other roots stuck on the unit circle?

Yes + same method, but useless since all \( \gamma \)s are non-reciprocal. And by C. Smyth’s Theorem we know that

\[
M(P) = M(A)M(B)M(C) = M(C) = M(\gamma) \geq 1.32 \ldots \text{smallest Pisot number}
\]

since the \( \gamma \)s are non-reciprocal.
Q3 : Is it possible to solve the Problem of Lehmer for the set of real reciprocal algebraic integers $\beta > 1$ by extending the method and using the above curves formed by the lenticuli of roots of the $P$s of $C$ in $\Re > 1/2$, or an angular part of them?

Yes + same method + completion of the lenticular curves using Rényi-Parry dynamical systems of numeration + rewriting trails + identification with Kala-Vavra’s Theorem.
Lecture 4.II:

- Solution of the Problem of Lehmer, via lenticuli of roots
- Rényi dynamical systems of numeration
- Completion of the lenticular curve, rewriting trails
- Identification of lenticular roots, Theorem of Kala - Vavra
- Dobrowolski type minoration, minoration of the Mahler measure.
Let $\beta > 1$ be a real number and let $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil \}$. If $\beta$ is not an integer, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$.

Let $x$ be a real number in the interval $[0, 1]$. A representation in base $\beta$ (or a $\beta$-representation; or a $\beta$-ary representation if $\beta$ is an integer) of $x$ is an infinite word $(x_i)_{i \geq 1}$ of $\mathcal{A}_\beta^\mathbb{N}$ such that

$$x = \sum_{i \geq 1} x_i \beta^{-i}.$$

The main difference with the case where $\beta$ is an integer is that $x$ may have several representations.
A particular $\beta$-representation, called the $\beta$-expansion, or the greedy $\beta$-expansion, and denoted by $d_\beta(x)$, of $x$ can be computed either by the greedy algorithm, or equivalently by the $\beta$-transformation

\[ T_\beta : x \mapsto \beta x \pmod{1} = \{\beta x\}. \]

The dynamical system $([0,1], T_\beta)$ is called the Rényi-Parry numeration system in base $\beta$, the iterates of $T_\beta$ providing the successive digits $x_i$ of $d_\beta(x)$. Denoting $T_\beta^0 := \text{Id}$, $T_\beta^1 := T_\beta$, $T_\beta^i := T_\beta(T_\beta^{i-1})$ for all $i \geq 1$, we have:

\[ d_\beta(x) = (x_i)_{i \geq 1} \quad \text{if and only if} \quad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor \]

and we write the $\beta$-expansion of $x$ as

\[ x = \cdot x_1 x_2 x_3 \ldots \quad \text{instead of} \quad x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \ldots \quad (1) \]

The digits are $x_1 = \lfloor \beta x \rfloor$, $x_2 = \lfloor \beta \{\beta x\} \rfloor$, $x_3 = \lfloor \beta \{\beta \{\beta x\}\} \rfloor$, \ldots , depend upon $\beta$. 

The Rényi-Parry numeration dynamical system in base $\beta$ allows the coding, as a (positional) $\beta$-expansion, of any real number $x$.

Indeed, if $x > 0$, there exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Hence $1/\beta \leq x/\beta^{k+1} < 1$; thus it is enough to deal with representations and $\beta$-expansions of numbers in the interval $[1/\beta, 1]$. In the case where $k \geq 1$, the $\beta$-expansion of $x$ is

$$x = x_1 x_2 \ldots x_k \cdot x_{k+1} x_{k+2} \ldots,$$

with $x_1 = \lfloor \beta(x/\beta^{k+1}) \rfloor$, $x_2 = \lfloor \beta\{\beta(x/\beta^{k+1})\} \rfloor$, $x_3 = \lfloor \beta\{\beta\{\beta(x/\beta^{k+1})\}\} \rfloor$, etc.

If $x < 0$, by definition: $d_\beta(x) = -d_\beta(-x)$.

The part $x_1 x_2 \ldots x_k$ is called the $\beta$-integer part of the $\beta$-expansion of $x$, and the terminant $\cdot x_{k+1} x_{k+2} \ldots$ is called the $\beta$-fractional part of $d_\beta(x)$. 
The set $\mathcal{A}_\beta^\mathbb{N}$ is endowed with the lexicographical order (not usual in number theory), and the product topology. The one-sided shift $\sigma: (x_i)_{i \geq 1} \mapsto (x_{i+1})_{i \geq 1}$ leaves invariant the subset $D_\beta$ of the $\beta$-expansions of real numbers in $[0,1)$. The closure of $D_\beta$ in $\mathcal{A}_\beta^\mathbb{N}$ is called the $\beta$-shift, and is denoted by $S_\beta$. The $\beta$-shift is a subshift of $\mathcal{A}_\beta^\mathbb{N}$, for which

$$d_\beta \circ T_\beta = \sigma \circ d_\beta$$

holds on the interval $[0,1]$. In other terms, $S_\beta$ is such that

$$x \in [0,1] \iff (x_i)_{i \geq 1} \in S_\beta$$

(2)

is bijective.

This one-to-one correspondence between the totally ordered interval $[0,1]$ and the totally lexicographically ordered $\beta$-shift $S_\beta$ is fundamental.
Parry has shown that only one sequence of digits entirely controls the $\beta$-shift $S_\beta$, and that the ordering is preserved when dealing with the greedy $\beta$-expansions.

Let us make precise how the usual inequality “$<$” on the real line is transformed into the inequality “$<_{lex}$”, meaning “lexicographically smaller with all its shifts”.

**The greatest element of** $S_\beta$ : it comes from $x = 1$ and is given either by the Rényi $\beta$-expansion of 1, or by a slight modification of it in case of finiteness. Let us make it precise. The greedy $\beta$-expansion of 1 is by definition denoted by

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots$$

and uniquely corresponds to

$$1 = \sum_{i=1}^{+\infty} t_i \beta^{-i},$$

where

$$t_1 = \lfloor \beta \rfloor, \quad t_2 = \lfloor \beta \{ \beta \} \rfloor = \lfloor \beta T_\beta(1) \rfloor, \quad t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor = \lfloor \beta T^2_\beta(1) \rfloor, \ldots$$
The sequence \((t_i)_{i \geq 1}\) is given by the orbit of one \((T^j_\beta(1))_{j \geq 0}\) by

\[
T^0_\beta(1) = 1, \quad T^j_\beta(1) = \beta^j - t_1 \beta^{j-1} - t_2 \beta^{j-2} - \ldots - t_j \in \mathbb{Z}[\beta] \cap [0, 1]
\]

for all \(j \geq 1\). The digits \(t_i\) belong to \(A_\beta\). We say that \(d_\beta(1)\) is finite if it ends in infinitely many zeros.

**Definition**

If \(d_\beta(1)\) is finite or ultimately periodic (i.e. eventually periodic), then the real number \(\beta > 1\) is said to be a *Parry number*. In particular, a Parry number \(\beta\) is said to be *simple* if \(d_\beta(1)\) is finite.

**Proposition (Parry)**

*The set of simple Parry numbers is dense in \((1, +\infty)\).*
From \((t_i)_{i \geq 1} \in \mathcal{A}_\beta^N\) is built \((c_i)_{i \geq 1} \in \mathcal{A}_\beta^N\), defined by

\[
c_1 c_2 c_3 \ldots := \begin{cases} 
  t_1 t_2 t_3 \ldots & \text{if } d_\beta(1) = 0.t_1 t_2 \ldots \text{ is infinite}, \\
  (t_1 t_2 \ldots t_{q-1}(t_q - 1))^\omega & \text{if } d_\beta(1) \text{ is finite},
\end{cases}
\]

where \((\cdot)^\omega\) means that the word within (\(\cdot\)) is indefinitely repeated. The sequence \((c_i)_{i \geq 1}\) is the unique element of \(\mathcal{A}_\beta^N\) which allows to obtain all the admissible \(\beta\)-expansions of all the elements of \([0, 1)\).
Definition (Conditions of Parry)
A sequence \((y_i)_{i \geq 0}\) of elements of \(A_\beta\) (finite or not) is said admissible if
\[
\sigma^j(y_0, y_1, y_2, \ldots) = (y_j, y_{j+1}, y_{j+2}, \ldots) <_{\text{lex}} (c_1, c_2, c_3, \ldots) \quad \text{for all } j \geq 0, \tag{6}
\]
where \(<_{\text{lex}}\) means lexicographically smaller.

Definition
A sequence \((a_i)_{i \geq 0} \in A_\beta^\mathbb{N}\) satisfying (7) is said to be Lyndon (or self-admissible):
\[
\sigma^n(a_0, a_1, a_2, \ldots) = (a_n, a_{n+1}, a_{n+2}, \ldots) <_{\text{lex}} (a_0, a_1, a_2, \ldots) \quad \text{for all } n \geq 1. \tag{7}
\]
The terminology comes from the introduction of such words by Lyndon, in honour of his work.

Any admissible representation \((x_i)_{i \geq 1} \in \mathcal{A}_\beta^\mathbb{N}\) corresponds, by (1), to a real number \(x \in [0, 1)\) and conversely the greedy \(\beta\)-expansion of \(x\) is \((x_i)_{i \geq 1}\) itself.

For an infinite admissible sequence \((y_i)_{i \geq 0}\) of elements of \(\mathcal{A}_\beta\) the (strict) lexicographical inequalities (6) constitute an infinite number of inequalities which are unusual in number theory.
In number theory, inequalities are often associated to collections of half-spaces in euclidean or adelic Geometry of Numbers (Minkowski’s Theorem, etc).

The conditions of Parry are of totally different nature since they refer to a reasonable control, order-preserving, of the gappiness (lacunarity) of the coefficient vectors of the power series.

In the correspondence

\[ [0, 1] \longleftrightarrow S_{\beta}, \]

the element \( x = 1 \) admits the maximal element \( d_\beta(1) \) as counterpart. The uniqueness of the \( \beta \)-expansion \( d_\beta(1) \) and its property to be Lyndon characterize the base of numeration \( \beta \) as follows.
Proposition (Parry)

Let \((a_0, a_1, a_2, \ldots)\) be a sequence of non-negative integers where \(a_0 \geq 1\) and \(a_n \leq a_0\) for all \(n \geq 0\). The unique solution \(\beta > 1\) of

\[
1 = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3} + \ldots
\]  

(8)

is such that \(d_\beta(1) = 0.a_0 a_1 a_2 \ldots\) if and only if

\[
\sigma^n(a_0, a_1, a_2, \ldots) = (a_n, a_{n+1}, a_{n+2}, \ldots) <_{lex} (a_0, a_1, a_2, \ldots) \quad \text{for all } n \geq 1. 
\]  

(9)
Theorem (VG, ’05)

Let $\beta > 1$ be an algebraic number such that $d_\beta(1)$ is infinite and gappy in the sense that there exist two infinite sequences $\{m_n\}_{n \geq 1}$ and $\{s_n\}_{n \geq 0}$ such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0$, $t_{s_n} \neq 0$ and $t_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$. Then

$$\limsup_{n \to +\infty} \frac{s_n}{m_n} \leq \frac{\Log(M(\beta))}{\Log \beta} \quad (10)$$
Varying the base of numeration $\beta$ in the interval $(1, 2)$:

for all $\beta \in (1, 2)$, being an algebraic number or a transcendental number, the alphabet $\mathcal{A}_\beta$ of the $\beta$-shift is always the same: $\{0, 1\}$. All the digits of all $\beta$-expansions $d_\beta(1)$ are zeroes or ones. Parry (’60) has proved that the relation of order $1 < \alpha < \beta < 2$ is preserved on the corresponding greedy $\alpha$- and $\beta$-expansions $d_\alpha(1)$ and $d_\beta(1)$ as follows.
Proposition

Let $\alpha > 1$ and $\beta > 1$. If the Rényi $\alpha$-expansion of 1 is

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \ldots,$$

i.e. $1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \ldots$

and the Rényi $\beta$-expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots,$$

i.e. $1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \ldots,$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \ldots) <_{\text{lex}} (t_1, t_2, t_3, \ldots)$. 
This interval is partitioned by the decreasing sequence \((\theta_n^{-1})_{n \geq 2}\) as

\[
(1, \frac{1+\sqrt{5}}{2}] = \bigcup_{n=2}^{\infty} \left[ \theta_{n+1}^{-1}, \theta_n^{-1} \right) \cup \{\theta_2^{-1}\}. \tag{11}
\]

Recall : \(\theta_n\) is the unique root of \(-1 + x + x^n\) in \((0, 1)\).
The condition of minimality on the length of the gaps of zeroes in \((t_i)_{i \geq 1}\) is only a function of the interval \([\theta_{n+1}^{-1}, \theta_n^{-1}]\) to which \(\beta\) belongs, when \(\beta\) tends to 1.

**Theorem**

Let \(n \geq 2\). A real number \(\beta \in (1, \frac{1+\sqrt{5}}{2}]\) belongs to \([\theta_{n+1}^{-1}, \theta_n^{-1}]\) if and only if the Rényi \(\beta\)-expansion of unity is of the form

\[
d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3} \ldots,
\]

with \(n_k \geq n - 1\) for all \(k \geq 1\).
Démonstration.

Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, Proposition 3 implies that the condition is sufficient. It is also necessary: $d_\beta(1)$ begins as $0.10^{n-1}1$ for all $\beta$ such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such $\beta$'s we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathcal{A}_\beta = \{0, 1\}$ common to all $\beta$'s, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2} \ldots$$

and $h_0, n_1, h_1, n_2, h_2, \ldots$ integers $\geq 0$. The self-admissibility lexicographic condition (9) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_3}, \ldots)$, which characterizes uniquely the base of numeration $\beta$, readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$. \(\square\)
Definition

Let $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ be a real number. The integer $n \geq 3$ such that 
$\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ is called the dynamical degree of $\beta$, and is denoted by $\text{dyg}(\beta)$. By convention we put : $\text{dyg}(\frac{1+\sqrt{5}}{2}) = 2$.

The function $n = \text{dyg}(\beta)$ is locally constant on the interval $(1, \frac{1+\sqrt{5}}{2}]$, is decreasing, takes all values in $\mathbb{N} \setminus \{0, 1\}$, and satisfies : $\lim_{\beta \searrow 1, \beta \to 1} \text{dyg}(\beta) = +\infty$. 
Let us observe that the equality \( \text{deg}(\beta) = \text{dyg}(\beta) = 2 \) holds if \( \beta = \frac{1+\sqrt{5}}{2} \), but the equality case is not the case in general.

**Definition**

A power series \( \sum_{j=0}^{+\infty} a_j z^j \), with \( a_j \in \{0, 1\} \) for all \( j \geq 0 \), is said to be **Lyndon (or self-admissible)** if its coefficient vector \( (a_i)_{i \geq 0} \) is Lyndon.

**Definition**

Let \( \beta \in (1, (1 + \sqrt{5})/2] \) be a real number, and \( d_\beta(1) = 0.t_1 t_2 t_3 \ldots \) its Rényi \( \beta \)-expansion of 1. The power series \( f_\beta(z) := -1 + \sum_{i \geq 1} t_i z^i \) is called the **Parry Upper function** at \( \beta \).
Proposition

For $1 < \beta < (1 + \sqrt{5})/2$ any real number, with $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$, the Parry Upper function $f_\beta(z)$ is such that $f_\beta(1/\beta) = 0$. It is such that $f_\beta(z) + 1$ has coefficients in the alphabet $\mathcal{A}_\beta = \{0, 1\}$ and is Lyndon. It takes the form

$$f_\beta(z) = G_{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + \ldots + z^{m_q} + z^{m_{q+1}} + \ldots$$  \hspace{1cm} (13)

with $m_1 - \text{dyg}(\beta) \geq \text{dyg}(\beta) - 1$, $m_{q+1} - m_q \geq \text{dyg}(\beta) - 1$ for $q \geq 1$. Conversely, given a power series

$$-1 + z + z^n + z^{m_1} + z^{m_2} + \ldots + z^{m_q} + z^{m_{q+1}} + \ldots$$  \hspace{1cm} (14)

with $n \geq 3$, $m_1 - n \geq n - 1$, $m_{q+1} - m_q \geq n - 1$ for $q \geq 1$, then there exists an unique $\beta \in (1, (1 + \sqrt{5})/2)$ for which $n = \text{dyg}(\beta)$ with $f_\beta(z)$ equal to (14).

Moreover, if $\beta$, $1 < \beta < (1 + \sqrt{5})/2$, is a reciprocal algebraic integer, the power series (13) is never a polynomial.
Completing the lenticular curve

Proposition

The class \( \mathcal{C} \) is the set of Parry Upper functions \( f_\beta(z) \) for all simple Parry numbers in \((1, (1 + \sqrt{5})/2)\).

Recall: the set of Parry numbers is dense in \((1, +\infty)\).

By the properties of \((x, \beta) \rightarrow T_\beta(x)\),

Proposition

The root functions of \( f_\beta(z) \) valued in \(|z| < 1\) are all continuous, as functions of \( \beta \in (1, \theta_2^{-1}) \setminus \bigcup_{n \geq 3} \{\theta_n^{-1}\} \).
Rewriting trails

Now consider a reciprocal algebraic integer

$$\beta \in (1, (1 + \sqrt{5})/2).$$

Two functions characterize the same “object” $\beta$ :

$$P_\beta(x) \quad \text{minimal polynomial}$$

and

$$f_\beta(z).$$

A priori they have nothing in common. The Parry Upper function has a lenticulus of zeroes containing $z = 1/\beta$. By a rewriting process, in base $\beta$, we show that, in a certain angular sector, approximately $(-\pi/18, +\pi/18)$, the lenticular zeroes should also be zeroes of the minimal polynomial $P_\beta(x)$.

The minorant of $\mathcal{M}(\beta)$ we are looking for arises from this subset of Galois conjugates of $1/\beta$, (therefore of $\beta$).
What is a rewriting trail?

Let us construct the rewriting trail from “$S_s$” (a section of $f_\beta(z)$) to “$P_\beta$”, at $\gamma_s^{-1}$.

The starting point is the identity $1 = 1$, to which we add $0 = S_{\gamma_s}(\gamma_s^{-1})$ in the (rhs) right hand side. Then we define the rewriting trail from the Rényi $\gamma_s^{-1}$-expansion of $1$

$$1 = 1 + S_{\gamma_s}(\gamma_s^{-1}) = t_1 \gamma_s^{-1} + t_2 \gamma_s^{-2} + \ldots + t_{s-1} \gamma_s^{-(s-1)} + t_s \gamma_s^{-s}$$  \hspace{1cm} (15)

(with $t_1 = 1, t_2 = t_3 = \ldots = t_{n-1} = 0, t_n = 1$, etc) to

$$- a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} + \ldots - a_{d-1} \gamma_s^{-(d-1)} - \gamma_s^{-d} = 1 - P_\beta(\gamma_s^{-1}),$$  \hspace{1cm} (16)
by “restoring” the digits of $1 - P_{\beta}(X)$ one after the other, from the left. We obtain a sequence $(A'_q(X))_{q \geq 1}$ of rewriting polynomials involved in this rewriting trail; for $q \geq 1$, $A'_q \in \mathbb{Z}[X]$, $\deg(A'_q) \leq q$ and $A'_q(0) = 1$. At the first step we add $0 = -(-a_1 - t_1)\gamma_s^{-1}S^*_\gamma_s(\gamma_s^{-1})$; and we obtain

$$1 = -a_1\gamma_s^{-1}$$

$$+(-(-a_1 - t_1)t_1 + t_2)\gamma_s^{-2} + (-(-a_1 - t_1)t_2 + t_3)\gamma_s^{-3} + \ldots$$

so that the height of the polynomial

$$(-(-a_1 - t_1)t_1 + t_2)X^2 + (-(-a_1 - t_1)t_2 + t_3)X^3 + \ldots$$

is $\leq H + 2$. 
At the second step we add \(0 = -(-a_2 - (-(-a_1 - t_1)t_1 + t_2)) \gamma_s^{-2} S_{\gamma_s}^{\ast} (\gamma_s^{-1}).\) Then we obtain

\[
1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} - [(−a_2 - (−(−a_1 − t_1)t_1 + t_2))t_1 + (−(−a_1 − t_1)t_2 + t_3)] \gamma_s^{-3} + \ldots
\]

where the height of the polynomial

\[
-[(−a_2 - (−(−a_1 − t_1)t_1 + t_2))t_1 + (−(−a_1 − t_1)t_2 + t_3)]X^3 + \ldots
\]

is \(\leq H + (H + 2) + (H + 2) = 3H + 4.\) Iterating this process \(d\) times we obtain

\[
1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} - \ldots - a_d \gamma_s^{-d} + \text{polynomial remainder in } \gamma_s^{-1}.
\]
Denote by $V(\gamma_s^{-1})$ this polynomial remainder in $\gamma_s^{-1}$, for some $V(X) \in \mathbb{Z}[X]$, and $X$ specializing in $\gamma_s^{-1}$. If we denote the upper bound of the height of the polynomial remainder $V(X)$, at step $q$, by $\lambda_q H + v_q$, we readily deduce: $v_q = 2^q$, and $\lambda_{q+1} = 2\lambda_q + 1$, $q \geq 1$, with $\lambda_1 = 1$; then $\lambda_q = 2^q - 1$.

To summarize, the first rewriting polynomials of the sequence $(A'_q(X))_{q \geq 1}$ involved in this rewriting trail are

\[ A'_1(X) = -1 - (-a_1 - t_1)X, \]
\[ A'_2(X) = -1 - (-a_1 - t_1)X - (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2, \quad \text{etc.} \]
For \( q \geq \deg(P_\beta) \), all the coefficients of \( P_\beta \) are “restored”; denote by 
\[(h_{q,j})_{j=0,1,\ldots,s-1}\]
the \( s \)-tuple of integers produced by this rewriting trail, at step \( q \). It is such that

\[
A'_q(\gamma^{-1})S^*_s(\gamma^{-1}) = -P(\gamma^{-1}) + \gamma^{-q-1}\left(\sum_{j=0}^{s-1} h_{q,j}\gamma^{-j}\right).
\]  

(17)

Then take \( q = d \). The (lhs) left-and side of (17) is equal to 0. Thus

\[
P(\gamma^{-1}) = \gamma^{-d-1}\left(\sum_{j=0}^{s-1} h_{d,j}\gamma^{-j}\right) \quad \implies \quad P(\gamma) = \sum_{j=0}^{s-1} h_{d,j}\gamma^{-j-1}.
\]
The height of the polynomial

\[ W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1} \]

is \( \leq (2^d - 1) H + 2^d \), \( \text{(18)} \)

and is independent of \( s \geq W_\nu \).

For any \( s \geq W_\nu \), let us observe that \( -P_\beta (\gamma_s^{-1}) \) is \( > 0 \), and that the sequence \( (\gamma_s^{-1})_s \) is decreasing. By an easy Lemma, the polynomial function \( x \rightarrow P_\beta (x) \) is positive on \( (0, \beta^{-1}) \), vanishes at \( \beta^{-1} \), and changes its sign for \( x > \beta^{-1} \), so that \( P_\beta (\gamma_s^{-1}) < 0 \). We have : \( \lim_{s \rightarrow \infty} P_\beta (\gamma_s^{-1}) = P_\beta (\beta^{-1}) = 0 \).
to allow Galois conjugation of $1/\beta$ we need to control the remaining sums after the rewriting trails.

This is made possible using Kala-Vara’s Theorem, and the fact that the irreducible factors $C(x)$, in the factorization of any $P \in \mathbb{C}$, never vanish on the unit circle.
Let us recall the definitions. The \((\delta, \mathcal{A})\)-representations for a given \(\delta \in \mathbb{C}\), \(|\delta| > 1\) and a given alphabet \(\mathcal{A} \subset \mathbb{C}\) finite, are expressions of the form \(\Sigma_{k \geq -L} a_k \delta^{-k}\), \(a_k \in \mathcal{A}\), for some integer \(L\). We denote

\[\text{Per}_\mathcal{A}(\delta) := \{x \in \mathbb{C} : x \text{ has an eventually periodic } (\delta, \mathcal{A})-\text{representation}\}.\]

Theorem (Kala - Vavra)

Let \(\delta \in \mathbb{C}\) be an algebraic number of degree \(d\), \(|\delta| > 1\), and \(a_d x^d - a_{d-1} x^{d-1} - \ldots - a_1 x - a_0 \in \mathbb{Z}[x]\), \(a_0 a_d \neq 0\), be its minimal polynomial. Suppose that \(|\delta'| \neq 1\) for any conjugate \(\delta'\) of \(\delta\), Then there exists a finite alphabet \(\mathcal{A} \subset \mathbb{Z}\) such that

\[Q(\delta) = \text{Per}_\mathcal{A}(\delta).\]
Dobrowolski-type minoration

Denote by \( a_{\text{max}} = 5.87433 \ldots \) the abscissa of the maximum of the function \( a \to \kappa(1, a) := \frac{1 - \exp\left( -\frac{\pi}{a} \right)}{2 \exp\left( \frac{\pi}{a} \right) - 1} \) on \((0, \infty)\). Let \( \kappa := \kappa(1, a_{\text{max}}) = 0.171573 \ldots \) be the value of the maximum. Let \( S := 2 \arcsin(\kappa/2) = 0.171784 \ldots \). Denote

\[
\Lambda_r \mu_r := \exp\left( \frac{-1}{\pi} \int_0^S \log \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] \, dx \right)
\]

\[
= 1.15411 \ldots, \quad \text{a value slightly below Lehmer’s number 1.17628\ldots} \quad (19)
\]

Theorem (Dobrowolski type minoration)

Let \( \alpha \) be a nonzero reciprocal algebraic integer which is not a root of unity such that \( \deg(\alpha) \geq 260 \), with \( M(\alpha) < 1.176280 \ldots \). Then

\[
M(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left( \frac{1}{\log(\deg(\alpha))} \right) \quad (20)
\]
Comparatively, in 1979, Dobrowolski, using an auxiliary function, obtained the asymptotic minoration, with \( n = \deg(\alpha) \),

\[
M(\alpha) > 1 + (1 - \varepsilon) \left( \frac{\Log \Log n}{\Log n} \right)^3, \quad n > n_0,
\]

with \( 1 - \varepsilon \) replaced by \( 1/1200 \) for \( n \geq 2 \), for an effective version of the minoration. In the inequality, the constant in the minorant is not any more 1 but 1.15411\ldots and the sign of the \( n \)-dependent term is negative, with an appreciable gain of \((\Log n)^2\) in the denominator.

It provides the non-trivial universal minorant of \( M \). But we do not know if Lehmer’s number 1.176280\ldots is the smallest Mahler measure. It is the smallest one known.