In this lecture we show that an **universal non-trivial minorant of** $\mathrm{M}(\alpha)$ for α a reciprocal nonzero real > 1 algebraic integer which is not root of unity can be obtained from the class of integer polynomials

$$\mathscr{C} := \{-1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} :$$

$$n \geq 3, m_1 - n \geq n-1, m_q - m_{q-1} \geq n-1 \quad \text{for} \quad 2 \leq q \leq s\}.$$

More precisely : from the curve formed by the lenticular roots of the Ps of $\mathscr{C}$.

Recall : for $P \in \mathscr{C}$, say $P(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s}$, with

$$n \geq 3, m_1 - n \geq n-1, m_q - m_{q-1} \geq n-1 \quad \text{for} \quad 2 \leq q \leq s,$$

$$P(x) = A(x)B(x)C(x)$$

with

- $A(x)$ product of cyclotomic polynomials, often trivial (75 %),

- $B(x)$ product of reciprocal non-cyclotomic irreducible factors, conjectured to be inexistant,

- $C(x)$ irreducible non-reciprocal (unique factor), vanishing at the unique zero of P in $(0,1)$.

Let γ be the unique zero of $P(x)$ in $(0,1)$: $C(\gamma) = 0$.

The lenticulus of roots of P is composed of Galois conjugates of C, conjugated with γ, and γ is non-reciprocal. Existence domain : $\Re(z) > 1/2$.

Example with $n = 37$:

$$P(x) := (-1 + x + x^{37}) + x^{81} + x^{140} + x^{184} + x^{232} + x^{285} + x^{350} + x^{389} + x^{450} + x^{514} + x^{550} + x^{590} + x^{649}$$

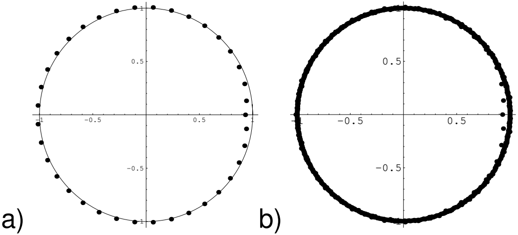

FIGURE: a) The 37 zeroes of $G_{37}(x) = -1 + x + x^{37}$, b) The 649 zeroes of $P(x) = G_{37}(x) + \ldots + x^{649}$. The lenticulus of roots of P is obtained by a very slight deformation of the restriction of the lenticulus of roots of G_{37} to the angular sector $|\arg z| < \pi/18$, off the unit circle. The other roots (nonlenticular) of P can be found in a narrow annular neighbourhood of $|z| = 1$.

Q1 : Is it possible to solve the Problem of Lehmer for the subcollection of $\mathscr{C}$ of the trinomials $(-1+x+x^n)_{n\geq 3}$ using the lenticuli of roots in $\Re > 1/2$?

Yes + method asymptotic expansions of the roots. Limit Mahler measure = 1.38....

Q2 : Is it possible to solve the Problem of Lehmer for the complete collection $\mathscr{C}$ of polynomials $(P)_{n,m_1,m_2,\ldots,m_s}$ using the curves formed by their lenticuli of roots in $\Re > 1/2$, or an angular part of them, without taking into account the other roots stuck on the unit circle ?

Yes + same method, but useless since all γs are non-reciprocal. And by C. Smyth's Theorem we know that

$$\mathrm{M}(P) = \mathrm{M}(A)\mathrm{M}(B)\mathrm{M}(C) = \mathrm{M}(C) = \mathrm{M}(\gamma) \geq 1.32\ldots \text{smallest Pisot number}$$

since the γs are **non-reciprocal**.

Q3 : Is it possible to solve the Problem of Lehmer for the set of real **reciprocal** algebraic integers $\beta > 1$ by extending the method and using the above curves formed by the lenticuli of roots of the *P*s of $\mathscr{C}$ in $\Re > 1/2$, or an angular part of them ?

Yes + same method + completion of the lenticular curves using Rényi-Parry dynamical systems of numeration + rewriting trails + identification with Kala-Vavra's Theorem.

Lecture 4.II :

- Solution of the Problem of Lehmer, via lenticuli of roots

- Rényi dynamical systems of numeration

- Completion of the lenticular curve, rewriting trails

- Identification of lenticular roots, Theorem of Kala - Vavra

- Dobrowolski type minoration, minoration of the Mahler measure.

ooooooo

Let $\beta > 1$ be a real number and let $\mathscr{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. If β is not an integer, then $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$.

Let x be a real number in the interval $[0, 1]$. A representation in base β (or a β-representation ; or a β-ary representation if β is an integer) of x is an infinite word $(x_i)_{i \geq 1}$ of $\mathscr{A}_\beta^{\mathbb{N}}$ such that

$$x = \sum_{i \geq 1} x_i \beta^{-i} .$$

The main difference with the case where β is an integer is that x may have several representations.

A particular β-representation, called the β-expansion, or the greedy β-expansion, and denoted by $d_\beta(x)$, of x can be computed either by the greedy algorithm, or equivalently by the β-transformation

$$T_\beta : x \mapsto \beta x \pmod 1 = \{\beta x\}.$$

The dynamical system $([0,1], T_\beta)$ is called the Rényi-Parry numeration system in base β, the iterates of T_β providing the successive digits x_i of $d_\beta(x)$. Denoting $T_\beta^0 := \mathrm{Id}, T_\beta^1 := T_\beta, T_\beta^i := T_\beta(T_\beta^{i-1})$ for all $i \geq 1$, we have :

$$d_\beta(x) = (x_i)_{i \geq 1} \qquad \text{if and only if} \qquad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

and we write the β-expansion of x as

$$x = \cdot x_1 x_2 x_3 \ldots \qquad \text{instead of} \qquad x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \ldots . \tag{1}$$

The digits are $x_1 = \lfloor \beta x \rfloor$, $x_2 = \lfloor \beta\{\beta x\} \rfloor$, $x_3 = \lfloor \beta\{\beta\{\beta x\}\} \rfloor, \ldots$, depend upon β.

The Rényi-Parry numeration dynamical system in base β allows the coding, as a (positional) β-expansion, of any real number x.

Indeed, if $x > 0$, there exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Hence $1/\beta \leq x/\beta^{k+1} < 1$; thus it is enough to deal with representations and β-expansions of numbers in the interval $[1/\beta, 1]$. In the case where $k \geq 1$, the β-expansion of x is

$$x = x_1 x_2 \ldots x_k \cdot x_{k+1} x_{k+2} \ldots,$$

with $x_1 = \lfloor \beta(x/\beta^{k+1}) \rfloor$, $x_2 = \lfloor \beta\{\beta(x/\beta^{k+1})\} \rfloor$, $x_3 = \lfloor \beta\{\beta\{\beta(x/\beta^{k+1})\}\} \rfloor$, etc.

If $x < 0$, by definition : $d_\beta(x) = -d_\beta(-x)$.

The part $x_1 x_2 \ldots x_k$ is called the β-integer part of the β-expansion of x, and the terminant $\cdot x_{k+1} x_{k+2} \ldots$ is called the β-fractional part of $d_\beta(x)$.

The set $\mathscr{A}_\beta^{\mathbb{N}}$ is endowed with the lexicographical order (not usual in number theory), and the product topology. The one-sided shift $\sigma : (x_i)_{i\geq 1} \mapsto (x_{i+1})_{i\geq 1}$ leaves invariant the subset D_β of the β-expansions of real numbers in $[0,1)$. The closure of D_β in $\mathscr{A}_\beta^{\mathbb{N}}$ is called the β-shift, and is denoted by S_β. The β-shift is a subshift of $\mathscr{A}_\beta^{\mathbb{N}}$, for which

$$d_\beta \circ T_\beta = \sigma \circ d_\beta$$

holds on the interval $[0,1]$. In other terms, S_β is such that

$$x \in [0,1] \qquad \longleftrightarrow \qquad (x_i)_{i\geq 1} \in S_\beta \tag{2}$$

is bijective.

This one-to-one correspondence between the totally ordered interval $[0,1]$ and the totally lexicographically ordered β-shift S_β is fundamental.

Parry has shown that only one sequence of digits entirely controls the β-shift S_β, and that the ordering is preserved when dealing with the greedy β-expansions.

Let us make precise how the usual inequality "$<$" on the real line is transformed into the inequality "$<_{lex}$", meaning "lexicographically smaller with all its shifts".

The greatest element of S_β : it comes from $x = 1$ and is given either by the Rényi β-expansion of 1, or by a slight modification of it in case of finiteness. Let us make it precise. The greedy β-expansion of 1 is by definition denoted by

$$d_\beta(1) = 0.t_1 t_2 t_3 \dots \qquad \text{and uniquely corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}, \qquad (3)$$

where

$$t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \dots \qquad (4)$$

The sequence $(t_i)_{i\geq 1}$ is given by the orbit of one $(T_\beta^j(1))_{j\geq 0}$ by

$$T_\beta^0(1) = 1, \; T_\beta^j(1) = \beta^j - t_1\beta^{j-1} - t_2\beta^{j-2} - \ldots - t_j \in \mathbb{Z}[\beta] \cap [0,1] \qquad (5)$$

for all $j \geq 1$. The digits t_i belong to $\mathscr{A}_\beta$. We say that $d_\beta(1)$ is finite if it ends in infinitely many zeros.

Definition

If $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic), then the real number $\beta > 1$ is said to be a *Parry number*. In particular, a Parry number β is said to be *simple* if $d_\beta(1)$ is finite.

Proposition (Parry)

The set of simple Parry numbers is dense in $(1, +\infty)$.

From $(t_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$ is built $(c_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$, defined by

$$c_1 c_2 c_3 \ldots := \begin{cases} t_1 t_2 t_3 \ldots & \text{if } d_\beta(1) = 0.t_1 t_2 \ldots \text{ is infinite,} \\ (t_1 t_2 \ldots t_{q-1}(t_q - 1))^\omega & \text{if } d_\beta(1) \text{ is finite, } = 0.t_1 t_2 \ldots t_q, \end{cases}$$

where $(\,)^\omega$ means that the word within ( ) is indefinitely repeated. The sequence $(c_i)_{i\geq 1}$ is the unique element of $\mathscr{A}_\beta^{\mathbb{N}}$ which allows to obtain all the admissible β-expansions of all the elements of $[0, 1)$.

Definition (Conditions of Parry)

A sequence $(y_i)_{i\geq 0}$ of elements of $\mathscr{A}_\beta$ (finite or not) is said *admissible* if

$$\sigma^j(y_0, y_1, y_2, \ldots) = (y_j, y_{j+1}, y_{j+2}, \ldots) <_{lex} (c_1, c_2, c_3, \ldots) \quad \text{for all } j \geq 0, \qquad (6)$$

where $<_{lex}$ means *lexicographically smaller*.

Definition

A sequence $(a_i)_{i\geq 0} \in \mathscr{A}_\beta^{\mathbb{N}}$ satisfying (7) is said to be *Lyndon (or self-admissible)* :

$$\sigma^n(a_0, a_1, a_2, \ldots) = (a_n, a_{n+1}, a_{n+2}, \ldots) <_{lex} (a_0, a_1, a_2, \ldots) \qquad \text{for all } n \geq 1. \quad (7)$$

The terminology comes from the introduction of such words by Lyndon, in honour of his work.

Any admissible representation $(x_i)_{i\geq 1} \in \mathscr{A}_\beta^{\mathbb{N}}$ corresponds, by (1), to a real number $x \in [0,1)$ and conversely the greedy β-expansion of x is $(x_i)_{i\geq 1}$ itself.

For an infinite admissible sequence $(y_i)_{i\geq 0}$ of elements of $\mathscr{A}_\beta$ the (strict) lexicographical inequalities (6) constitute an infinite number of inequalities which are unusual in number theory.

In number theory, inequalities are often associated to collections of half-spaces in euclidean or adelic Geometry of Numbers (Minkowski's Theorem, etc).

The conditions of Parry are of totally different nature since they refer to a reasonable control, order-preserving, of the gappiness (lacunarity) of the coefficient vectors of the power series.

In the correspondence

$$[0,1] \longleftrightarrow S_{\beta},$$

the element $x = 1$ admits the maximal element $d_{\beta}(1)$ as counterpart. The uniqueness of the β-expansion $d_{\beta}(1)$ and its property to be Lyndon characterize the base of numeration β as follows.

Proposition (Parry)

Let $(a_0, a_1, a_2, \ldots)$ *be a sequence of non-negative integers where* $a_0 \geq 1$ *and* $a_n \leq a_0$ *for all* $n \geq 0$*. The unique solution* $\beta > 1$ *of*

$$1 = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3} + \ldots \tag{8}$$

is such that $d_\beta(1) = 0.a_0 a_1 a_2 \ldots$ *if and only if*

$$\sigma^n(a_0, a_1, a_2, \ldots) = (a_n, a_{n+1}, a_{n+2}, \ldots) <_{lex} (a_0, a_1, a_2, \ldots) \qquad \textit{for all } n \geq 1. \tag{9}$$

Theorem (VG, '05)

Let $\beta > 1$ be an algebraic number such that $d_\beta(1)$ is infinite and gappy in the sense that there exist two infinite sequences $\{m_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 0}$ such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \ldots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \ldots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0$, $t_{s_n} \neq 0$ and $t_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$. Then

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} \leq \frac{\mathrm{Log}\,(\mathrm{M}(\beta))}{\mathrm{Log}\,\beta} \tag{10}$$

Varying the base of numeration β *in the interval* $(1,2)$ *:*

for all $\beta \in (1,2)$, being an algebraic number or a transcendental number, the alphabet $\mathscr{A}_\beta$ of the β-shift is always the same : $\{0,1\}$. All the digits of all β-expansions $d_\beta(1)$ are zeroes or ones. Parry ('60) has proved that the relation of order $1 < \alpha < \beta < 2$ is preserved on the corresponding greedy α- and β- expansions $d_\alpha(1)$ and $d_\beta(1)$ as follows.

Proposition

Let $\alpha > 1$ and $\beta > 1$. If the Rényi α-expansion of 1 is

$$d_\alpha(1) = 0.t'_1 t'_2 t'_3 \ldots, \qquad i.e. \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \ldots$$

and the Rényi β-expansion of 1 is

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots, \qquad i.e. \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \ldots,$$

then $\alpha < \beta$ if and only if $(t'_1, t'_2, t'_3, \ldots) <_{lex} (t_1, t_2, t_3, \ldots)$.

This interval is partitioned by the decreasing sequence $(\theta_n^{-1})_{n \geq 2}$ as

$$\left(1, \frac{1+\sqrt{5}}{2}\right] = \bigcup_{n=2}^{\infty} \left[\theta_{n+1}^{-1}, \theta_n^{-1}\right) \ \bigcup \ \left\{\theta_2^{-1}\right\}. \tag{11}$$

Recall : θ_n is the unique root of $-1 + x + x^n$ in $(0, 1)$.

The condition of minimality on the length of the gaps of zeroes in $(t_i)_{i\geq 1}$ is only a function of the interval $\left[\theta_{n+1}^{-1}, \theta_n^{-1}\right)$ to which β belongs, when β tends to 1.

Theorem

Let $n \geq 2$. A real number $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ belongs to $[\theta_{n+1}^{-1}, \theta_n^{-1})$ if and only if the Rényi β-expansion of unity is of the form

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\ldots, \tag{12}$$

with $n_k \geq n-1$ for all $k \geq 1$.

Démonstration.

Since $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$ and $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$, Proposition 3 implies that the condition is sufficient. It is also necessary : $d_\beta(1)$ begins as $0.10^{n-1}1$ for all β such that $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$. For such βs we write $d_\beta(1) = 0.10^{n-1}1u$ with digits in the alphabet $\mathscr{A}_\beta = \{0,1\}$ common to all βs, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2}\ldots$$

and $h_0, n_1, h_1, n_2, h_2, \ldots$ integers ≥ 0. The self-admissibility lexicographic condition (9) applied to the sequence $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_3}, \ldots)$, which characterizes uniquely the base of numeration β, readily implies $h_0 = 0$ and $h_k = 1$ and $n_k \geq n-1$ for all $k \geq 1$. □

Definition

Let $\beta \in (1, \frac{1+\sqrt{5}}{2}]$ be a real number. The integer $n \geq 3$ such that $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ is called the *dynamical degree* of β, and is denoted by $\mathrm{dyg}(\beta)$. By convention we put : $\mathrm{dyg}(\frac{1+\sqrt{5}}{2}) = 2$.

The function $n = \mathrm{dyg}(\beta)$ is locally constant on the interval $(1, \frac{1+\sqrt{5}}{2}]$, is decreasing, takes all values in $\mathbb{N} \setminus \{0, 1\}$, and satisfies : $\lim_{\beta > 1, \beta \to 1} \mathrm{dyg}(\beta) = +\infty$.

Let us observe that the equality $\deg(\beta) = \mathrm{dyg}(\beta) = 2$ holds if $\beta = \frac{1+\sqrt{5}}{2}$, but the equality case is not the case in general.

Definition

A power series $\sum_{j=0}^{+\infty} a_j z^j$, with $a_j \in \{0, 1\}$ for all $j \geq 0$, is said to be *Lyndon (or self-admissible)* if its coefficient vector $(a_i)_{i \geq 0}$ is Lyndon.

Definition

Let $\beta \in (1, (1+\sqrt{5})/2]$ be a real number, and $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$ its Rényi β-expansion of 1. The power series $f_\beta(z) := -1 + \sum_{i \geq 1} t_i z^i$ is called the *Parry Upper function* at β.

Proposition

For $1 < \beta < (1+\sqrt{5})/2$ *any real number, with* $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$*, the Parry Upper function* $f_\beta(z)$ *is such that* $f_\beta(1/\beta) = 0$*. It is such that* $f_\beta(z)+1$ *has coefficients in the alphabet* $\mathscr{A}_\beta = \{0,1\}$ *and is Lyndon. It takes the form*

$$f_\beta(z) = G_{\mathrm{dyg}(\beta)} + z^{m_1} + z^{m_2} + \ldots + z^{m_q} + z^{m_{q+1}} + \ldots \qquad (13)$$

with $m_1 - \mathrm{dyg}(\beta) \geq \mathrm{dyg}(\beta) - 1$, $m_{q+1} - m_q \geq \mathrm{dyg}(\beta) - 1$ *for* $q \geq 1$*. Conversely, given a power series*

$$-1 + z + z^n + z^{m_1} + z^{m_2} + \ldots + z^{m_q} + z^{m_{q+1}} + \ldots \qquad (14)$$

with $n \geq 3$, $m_1 - n \geq n-1$, $m_{q+1} - m_q \geq n-1$ *for* $q \geq 1$*, then there exists an unique* $\beta \in (1, (1+\sqrt{5})/2)$ *for which* $n = \mathrm{dyg}(\beta)$ *with* $f_\beta(z)$ *equal to* (14)*.*

Moreover, if β*,* $1 < \beta < (1+\sqrt{5})/2$*, is a reciprocal algebraic integer, the power series* (13) *is never a polynomial.*

Proposition

The class $\mathscr{C}$ is the set of Parry Upper functions $f_\beta(z)$ for all simple Parry numbers in $(1, (1+\sqrt{5})/2)$.

Recall : the set of Parry numbers is dense in $(1, +\infty)$.

By the properties of $(x, \beta) \to T_\beta(x)$,

Proposition

The root functions of $f_\beta(z)$ valued in $|z| < 1$ are all continuous, as functions of $\beta \in (1, \theta_2^{-1}) \setminus \cup_{n \geq 3} \{\theta_n^{-1}\}$.

Rewriting trails

Now consider a reciprocal algebraic integer

$$\beta \in (1, (1+\sqrt{5})/2).$$

Two functions characterize the same "object" β :

$$P_\beta(x) \qquad \text{minimal polynomial}$$

and

$$f_\beta(z).$$

A priori they have nothing in common. The Parry Upper function has a lenticulus of zeroes containing $z = 1/\beta$. By a rewriting process, in base β, we show that, in a certain angular sector, approximately $(-\pi/18, +\pi/18)$, the lenticular zeroes should also be zeroes of the minimal polynomial $P_\beta(x)$.

The minorant of $\mathrm{M}(\beta)$ we are looking for arises from this subset of Galois conjugates of $1/\beta$, (therefore of β).

What is a rewriting trail ?

Let us construct the rewriting trail from "S_s" (a section of $f_\beta(z)$) to "P_β", at γ_s^{-1}.

The starting point is the identity $1 = 1$, to which we add $0 = S_{\gamma_s}(\gamma_s^{-1})$ in the (rhs) right hand side. Then we define the rewriting trail from the Rényi γ_s^{-1}-expansion of 1

$$1 = 1 + S_{\gamma_s}(\gamma_s^{-1}) = t_1 \gamma_s^{-1} + t_2 \gamma_s^{-2} + \ldots + t_{s-1} \gamma_s^{-(s-1)} + t_s \gamma_s^{-s} \tag{15}$$

(with $t_1 = 1, t_2 = t_3 = \ldots = t_{n-1} = 0, t_n = 1$, etc) to

$$-a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} + \ldots - a_{d-1} \gamma_s^{-(d-1)} - \gamma_s^{-d} = 1 - P_\beta(\gamma_s^{-1}), \tag{16}$$

by “restoring” the digits of $1-P_{\beta}(X)$ one after the other, from the left. We obtain a sequence $(A'_q(X))_{q\geq 1}$ of rewriting polynomials involved in this rewriting trail ; for $q\geq 1$, $A'_q\in\mathbb{Z}[X]$, $\deg(A'_q)\leq q$ and $A'_q(0)=1$. At the first step we add $0=-(-a_1-t_1)\gamma_s^{-1}S^*_{\gamma_s}(\gamma_s^{-1})$; and we obtain

$$1=-a_1\gamma_s^{-1}$$

$$+(-(-a_1-t_1)t_1+t_2)\gamma_s^{-2}+(-(-a_1-t_1)t_2+t_3)\gamma_s^{-3}+\ldots$$

so that the height of the polynomial

$$(-(-a_1-t_1)t_1+t_2)X^2+(-(-a_1-t_1)t_2+t_3)X^3+\ldots$$

is $\leq H+2$.

At the second step we add $0 = -(-a_2 - (-(-a_1 - t_1)t_1 + t_2))\gamma_s^{-2} S^*_{\gamma_s}(\gamma_s^{-1})$.
Then we obtain

$$1 = -a_1\gamma_s^{-1} - a_2\gamma_s^{-2}$$

$$-[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]\gamma_s^{-3} + \ldots$$

where the height of the polynomial

$$-[(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)]X^3 + \ldots$$

is $\leq H + (H+2) + (H+2) = 3H + 4$. Iterating this process d times we obtain

$$1 = -a_1\gamma_s^{-1} - a_2\gamma_s^{-2} - \ldots - a_d\gamma_s^{-d}$$

$$+ \ \textit{polynomial remainder in} \ \gamma_s^{-1}.$$

Denote by $V(\gamma_s^{-1})$ this polynomial remainder in γ_s^{-1}, for some $V(X) \in \mathbb{Z}[X]$, and X specializing in γ_s^{-1}. If we denote the upper bound of the height of the polynomial remainder $V(X)$, at step q, by $\lambda_q H + v_q$, we readily deduce : $v_q = 2^q$, and $\lambda_{q+1} = 2\lambda_q + 1$, $q \geq 1$, with $\lambda_1 = 1$; then $\lambda_q = 2^q - 1$.

To summarize, the first rewriting polynomials of the sequence $(A'_q(X))_{q\geq 1}$ involved in this rewriting trail are

$$A'_1(X) = -1 - (-a_1 - t_1)X,$$

$$A'_2(X) = -1 - (-a_1 - t_1)X - (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2, \quad \text{etc.}$$

For $q \geq \deg(P_\beta)$, all the coefficients of P_β are "restored" ; denote by $(h_{q,j})_{j=0,1,\ldots,s-1}$ the s-tuple of integers produced by this rewriting trail, at step q. It is such that

$$A'_q(\gamma_s^{-1}) S^*_{\gamma_s}(\gamma_s^{-1}) = -P(\gamma_s^{-1}) + \gamma_s^{-q-1} \Big(\sum_{j=0}^{s-1} h_{q,j} \gamma_s^{-j} \Big). \tag{17}$$

Then take $q = d$. The (lhs) left-and side of (17) is equal to 0. Thus

$$P(\gamma_s^{-1}) = \gamma_s^{-d-1} \Big(\sum_{j=0}^{s-1} h_{d,j} \gamma_s^{-j} \Big) \qquad \Longrightarrow \qquad P(\gamma_s) = \sum_{j=0}^{s-1} h_{d,j} \gamma_s^{-j-1}.$$

The height of the polynomial

$$W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1} \qquad \text{is} \qquad \leq (2^d - 1)H + 2^d, \tag{18}$$

and is independent of $s \geq W_v$.

For any $s \geq W_v$, let us observe that $-P_\beta(\gamma_s^{-1})$ is > 0, and that the sequence $(\gamma_s^{-1})_s$ is decreasing. By an easy Lemma, the polynomial function $x \to P_\beta(x)$ is positive on $(0, \beta^{-1})$, vanishes at β^{-1}, and changes its sign for $x > \beta^{-1}$, so that $P_\beta(\gamma_s^{-1}) < 0$. We have : $\lim_{s\to\infty} P_\beta(\gamma_s^{-1}) = P_\beta(\beta^{-1}) = 0$.

to allow Galois conjugation of $1/\beta$ we need to control the remaining sums after the rewriting trails.

This is made possible using Kala-Vara's Theorem, and the fact that the irreducible factors $C(x)$, in the factorization of any $P \in \mathscr{C}$, never vanish on the unit circle.

Let us recall the definitions. The $(\delta, \mathscr{A})$-representations for a given $\delta \in \mathbb{C}$, $|\delta| > 1$ and a given alphabet $\mathscr{A} \subset \mathbb{C}$ finite, are expressions of the form $\sum_{k \geq -L} a_k \delta^{-k}$, $a_k \in \mathscr{A}$, for some integer L. We denote

$$\mathrm{Per}_{\mathscr{A}}(\delta) := \{x \in \mathbb{C} : x \text{ has an eventually periodic } (\delta, \mathscr{A})\text{−representation}\}.$$

Theorem (Kala - Vavra)

Let $\delta \in \mathbb{C}$ be an algebraic number of degree d, $|\delta| > 1$, and $a_d x^d - a_{d-1} x^{d-1} - \ldots - a_1 x - a_0 \in \mathbb{Z}[x]$, $a_0 a_d \neq 0$, be its minimal polynomial. Suppose that $|\delta'| \neq 1$ for any conjugate δ' of δ, Then there exists a finite alphabet $\mathscr{A} \subset \mathbb{Z}$ such that

$$\mathbb{Q}(\delta) = \mathrm{Per}_{\mathscr{A}}(\delta).$$

Dobrowolski-type minoration

Denote by $a_{\max} = 5.87433\ldots$ the abscissa of the maximum of the function $a \to \kappa(1,a) := \frac{1-\exp(\frac{-\pi}{a})}{2\exp(\frac{\pi}{a})-1}$ on $(0,\infty)$. Let $\kappa := \kappa(1,a_{\max}) = 0.171573\ldots$ be the value of the maximum. Let $S := 2\arcsin(\kappa/2) = 0.171784\ldots$. Denote

$$\Lambda_r\mu_r := \exp\Big(\frac{-1}{\pi}\int_0^S \mathrm{Log}\Big[\frac{1+2\sin(\frac{x}{2})-\sqrt{1-12\sin(\frac{x}{2})+4(\sin(\frac{x}{2}))^2}}{4}\Big]dx\Big)$$

$$= 1.15411\ldots, \qquad \text{a value slightly below Lehmer's number } 1.17628\ldots \tag{19}$$

Theorem (Dobrowolski type minoration)

Let α be a nonzero reciprocal algebraic integer which is not a root of unity such that $\mathrm{dyg}(\alpha) \geq 260$, with $\mathrm{M}(\alpha) < 1.176280\ldots$. Then

$$\mathrm{M}(\alpha) \geq \Lambda_r\mu_r - \Lambda_r\mu_r\,\frac{S}{2\pi}\Big(\frac{1}{\mathrm{Log}\,(\mathrm{dyg}(\alpha))}\Big) \tag{20}$$

Comparatively, in 1979, Dobrowolski, using an auxiliary function, obtained the asymptotic minoration, with $n = \deg(\alpha)$,

$$\mathrm{M}(\alpha) > 1 + (1-\varepsilon)\left(\frac{\mathrm{Log}\,\mathrm{Log}\, n}{\mathrm{Log}\, n}\right)^3, \qquad n > n_0, \tag{21}$$

with $1-\varepsilon$ replaced by $1/1200$ for $n \geq 2$, for an effective version of the minoration. In the inequality, the constant in the minorant is not any more 1 but $1.15411\ldots$ and the sign of the n-dependent term is negative, with an appreciable gain of $(\mathrm{Log}\, n)^2$ in the denominator.

It provides the non-trivial universal minorant of M. But we do not know if Lehmer's number 1.176280... is the smallest Mahler measure. It is the smallest one known.