Journées/Days

Diophantine Equations and Algebraic Equations

May 2-3, 2023

Visio-Talks

LAMA - University Savoie Mont Blanc

Chambéry - Le Bourget-du-Lac

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INTRODUCTION

The main purpose of this workshop is to bring together PhD students and researchers who are interested in algebraic and/or Diophantine equations. In Part I, we first show how to apply the modular method to a class of generalized Fermat equations and certain generalized Lebesgue-Ramanujan-Nagell type equations. Secondly, introducing some known results on Schäffer's conjecture we will show that how to use Bernoulli numbers/polynomials for solving some Diophantine equations with power sums. In Part II we are interested in the set of zeroes of univariate lacunary polynomials in a special class of integer polynomials having coefficients in $\{0, 1\}$, except the constant term equal to -1, and a gappiness whose size is controlled a minima by an unique integer. We establish the link between this class of integer polynomials and a dense subset of non-reciprocal algebraic integers $\beta > 1$ close to 1, by the Rényi numeration dynamical system and the β-transformation. We show that the completion of this class of zeroes contains an analytic curve which contains Galois conjugates of the βs > 1 which are reciprocal algebraic integers close to 1. Some consequences on the minoration of the Mahler measure $\mathrm{M}(\beta)$ of such βs are evoked.

Part I: A great deal of number theory arises from the discussion of the integer or rational solutions of a polynomial equation with integer coefficients. Such equations are called Diophantine equations. In 1637, Pierre de Fermat conjectured that the equation $x^p + y^p = z^p$ has no solutions in non-zero integers x, y, z for $p \geq 3$. This is known as Pierre de Fermat's Last Theorem. In 1995, the most important progress in the field of the Diophantine equations has been with Andrew Wiles' proof of Pierre de Fermat's Last Theorem. His proof is based on deep results about Galois representations associated to elliptic curves and modular forms. The method of using such results to deal with Diophantine problems, is called the modular approach.

In the same century, Jakob Bernoulli (1655-1705) introduced the Bernoulli numbers in connection to the study of the sums of powers of consecutive integers $1^k + 2^k + \cdots + n^k$. The study of the polynomial Diophantine equation in the form of

$$1^k + 2^k + ... + x^k = y^n, \quad x, y \in \mathbb{Z}^+, \quad n \geq 2 \tag{0.1}$$

has been going on for more than a hundred years. In 1956, Juan Schäffer showed, for $k \geq 1$ and $n \geq 2$, that (0.1) possesses at most finitely many solutions in positive integers x and y, unless $(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\}$, where, in each case, there are infinitely many such solutions. Schäffer's conjectured that (0.1) has the unique non-trivial (i.e. $(x, y) \neq (1, 1)$) solution, namely $(k, n, x, y) = (2, 2, 24, 70)$. The correctness of this conjecture has been proved for some cases. But, it has not been proved completely yet.

Part II: The question of finding exact expressions for the roots of integer polynomials, say $\sum_{i=0}^{n} a_i x^i$, as a function of the degree n and the coefficient vector $(a_0, a_1, \ldots, a_n)$, and understanding the role of lacunarity on the factorization, has received a lot of attention. Let us mention the Renaissance mathematicians at Bologna, Fontana, Tschirnhaus, Leibniz, Lagrange, Abel, Galois, Liouville, for the method of radical expressions (multiplication, addition, subtraction, division, extraction of roots, only permitted on the coefficients). Galois theory says that the case $n = 5$, for the degree, is a critical case above which other techniques than radical expressions should be applied for solving the equations. Mellin (1915) had overcome this difficulty by solving general algebraic equations by suitable Cauchy integration in the complex plane using hypergeometric functions and the Γ-function (G. Belardinelli, Fonctions hypergéométriques de plusieurs variables et résolution analytique des équations algébriques générales, Mém. Sci. Math. Fasc. 145, Gauthiers-Villars, Paris, 1960). Later, the role of lacunarity on the factorization was investigated by Schinzel, Dobrowolski, in terms of cyclotomic, reciprocal noncyclotomic and nonreciprocal components, and generalized to several variables.

In this talk attention will be focused on a class of almost-Newman integer polynomials which are sums of a trinomial $-1 + x + x^n$, $n \geq 3$, and a perturbation Newman polynomial having distanciation between successive monomials greater than or equal to $n-1$. Though Mellin's approach be applicable, for all the roots, the method of asymptotic expansions of the roots of the trinomials $-1+x+x^n$ which is developed (in [7]) is more powerful to give access to more precise formulations and properties of a subcategory of roots, called lenticular. Consequences are evoked in [7] [8] [9] [10].

Speakers: - Gökhan SOYDAN (Bursa Uludag University, Turkey),
- Jean-Louis VERGER-GAUGRY (CNRS, University Savoie Mont-Blanc).

Part I: Lectures 1, 2, 3 by Gökhan SOYDAN,
Part II: Lecture 4 by Jean-Louis VERGER-GAUGRY.

PROGRAMME

Lecture No 1: On the solutions of a class of generalized Fermat equations of signature $(2, 2n, 3)$

(Tuesday May 2 - 9:15-11:15 + 14:00-15:00)

In this lecture, we first consider the Diophantine equation $7x^2 + y^{2n} = 4z^3$. We determine all solutions to this equation for $n = 2, 3, 4$ and 5. We formulate a Kraus type criterion for showing that the Diophantine equation $7x^2 + y^{2p} = 4z^3$ has no non-trivial proper integer solutions for specific primes $p > 7$. We computationally verify the criterion for all primes $7 < p < 10^9$, $p \neq 13$. We use the symplectic method and quadratic reciprocity to show that the Diophantine equation $7x^2+y^{2p} = 4z^3$ has no non-trivial proper solutions for a positive proportion of primes p.

Secondly, we consider the Diophantine equation $x^2 + 7y^{2n} = 4z^3$, determining all families of solutions for $n = 2$ and 3, as well as giving a (mostly) conjectural description of the solutions for $n = 4$ and primes $n \geq 5$.

This lecture is based on [2, 3].

Lecture No 2: On some Diophantine equations with power sums.

(Tuesday May 2 - 15:00-16:00 + Wednesday May 3 - 9:00-11:00)

Let $k, l \geq 2$ be fixed integers. In this lecture, we first consider the Diophantine equation

$$(x+1)^k + (x+2)^k + ... + (\ell x)^k = y^n, \quad x, y \in \mathbb{Z}, \quad n \geq 2. \tag{0.2}$$

Firstly, we prove that all solutions of the equation (0.2) in integers x, y, n with $x, y \geq 1, n \geq 2$ satisfy $n < C_1$ where $C_1 = C_1(l, k)$ is an effectively computable constant. Secondly, we prove that all solutions of this equation in integers x, y, n with $x, y \geq 1, n \geq 2$ and $k \neq 3$ satisfy $\max\{x, y, n\} < C_2$ where C_2 is an effectively computable constant depending only on k and l.

Next, we consider the Diophantine equation

$$(x-d)^2 + x^2 + (x+d)^2 = y^n. \tag{0.3}$$

Firstly, we give an explicit formula for all positive integer solutions of the equation (0.3) when n is an odd prime and $d = p^r$, $p > 3$ a prime. Secondly, under the assumption of our first result, we prove that the equation (0.3) has at most one solution (x, y). Thirdly, for a general d, we prove the following two results: (i) if every odd prime divisor q of d satisfies $q \not\equiv \pm 1 \pmod{2n}$, then (0.3) has only the solution $(x, y, d, n) = (21, 11, 2, 3)$. (ii) if $n > 228000$ and $d > 8\sqrt{2}$, then all solutions (x, y) of (0.3) satisfy $y^n < 2^{3/2}d^3$. This lecture is based on [1], [4] and [6].

Lecture No 3: On the solutions of some generalized Lebesgue-Ramanujan-Nagell equations

(Wednesday May 3 - 14:00-16:00)

In this lecture, we first introduce generalized Lebesgue-Ramanujan-Nagell equations. Then, using the modular approach, we show that if $k \equiv 0 \pmod 4$, $30 < k < 724$ and $2k-1$ is an odd prime power, then under the GRH (generalized Riemann hypothesis), the generalized Ramanujan-Nagell equation of the form $x^2 + (2k-1)^y = k^z$ has only one positive integer solution $(x, y, z) = (k-1, 1, 2)$. The above results solve some difficult cases of Terai's conecture concerning this equation. This lecture is based on [5].

Lecture No 4: Algebraic equations from a class of integer polynomials having lacunarity conditions

(Tuesday May 2 - 11:15-12:30 + Wednesday May 3 - 11:00- 12:30)

In this lecture we consider the class of integer polynomials

$$\mathcal{C} := \{-1 + x + x^n + x^{m_1} + x^{m_2} + \ldots + x^{m_s} : \\ n \geq 3, m_1 - n \geq n-1, m_q - m_{q-1} \geq n-1 \quad \text{for} \quad 2 \leq q \leq s\}.$$

Mellin's approach provides exact expressions of all the roots, for any $n \geq 3$ and any $s \geq 0$. The case $s = 0$ corresponds to the trinomials. We show that it is not sufficient to classify the roots, those close to the unit circle, those off the unit circle. We develop the method of the asymptotic expansions (called "Poincaré") for the solutions of $-1 + x + x^n = 0$, for $n \geq 3$, and extend it to any polynomial of $\mathcal{C}$. Those off the unit circle form lenticuli. The set of lenticular roots can be completed and forms a curve of solutions, image of a neighbourhood of 1 in $(0, 1)$. To prove this, we show that any polynomial P of $\mathcal{C}$ is uniquely associated to a Rényi numeration dynamical system $([0, 1], T_\beta)$ where T_β is the β-transformation and β^{-1} is the unique zero of P in $(0, 1)$. The factorization of any $P \in \mathcal{C}$ is deduced from Ljunggren's method and Kronecker's average value theorem. A Conjecture on the non-existence of reciprocal non-cyclotomic components is formulated. For the reciprocal algebraic integers $\beta > 1$ very close to 1, using Kala-Vavra's theorem, we show an accumulation of conjugates of β on the completed lenticular curve. A (Dobrowolski-type) minoration of the Mahler measure $\mathrm{M}(\beta)$ is deduced.

SCHEDULE (Paris France local time)

Tuesday May 2:

9:00 - 9:15 Welcome - Bienvenue - Introduction

09:15 - 11:15 Lecture 1: On the solutions of a class of generalized Fermat equations of signature $(2, 2n, 3)$.

11:15 - 12:30 Lecture 4: Algebraic equations from a class of integer polynomials having lacunarity conditions.

Break

14:00 - 15:00 Lecture 1 (+): On the solutions of a class of generalized Fermat equations of signature $(2, 2n, 3)$.

15:00 - 16:00 Lecture 2: On some Diophantine equations with power sums.

Wednesday May 3:

09:00 - 11:00 Lecture 2 (+): On some Diophantine equations with power sums.

11:00 - 12:30 Lecture 4 (+): Algebraic equations from a class of integer polynomials having lacunarity conditions.

Break

14:00 - 16:00 Lecture 3: On the solutions of some generalized Lebesgue-Ramanujan-Nagell equations.

References

[1] D. Bartoli and G. Soydan, The Diophantine equation $(x+1)^k + (x+2)^k + ... + (lx)^k = y^n$ revisited, *Publ. Math. Debrecen* **96/1-2** (2020), 111-120.

[2] K. Chałupka, A. Dabrowski and G. Soydan, On a class of generalized Fermat equations of signature $(2, 2n, 3)$, *J. Number Theory* **34** (2022), 154-178.

[3] K. Chałupka, A. Dabrowski and G. Soydan, On a class of generalized Fermat equations of signature $(2, 2n, 3)$, II, submitted.

[4] M.-H. Le and G. Soydan, On the power values of the sum of three squares in arithmetic progression, Math. Commun. 27 (2022), 137–150.

[5] E. K. Mutlu, M. H. Le and G. Soydan, A modular approach to the generalized Lebesgue-Ramanujan-Nagell equation, *Indagationes Mathematicae* **33** (2022), 992–1000.

[6] G. Soydan, On the Diophantine equation $(x+1)^k + (x+2)^k + ... + (lx)^k = y^n$, *Publ. Math. Debrecen* **91** (2017), 369-382.

[7] J.-L. Verger-Gaugry, On the Conjecture of Lehmer, limit Mahler measure of trinomials and asymptotic expansions, Uniform Distribution Theory J. **11** (2016), 79–139.

[8] D. Dutykh, J.-L. Verger-Gaugry, Alphabets, rewriting trails, periodic representation in algebraic basis, Res. Number Theory 7:64 (2021).

[9] J.-L. Verger-Gaugry, A proof of the Conjecture of Lehmer

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[10] D. Dutykh, J.-L. Verger-Gaugry, On a class of lacunary almost-Newman polynomials modulo p and density theorems, Unif. Distrib. Theory 17, No 1 (2022), 29–54.

The modular approach :

[11] S. Siksek, The Modular Approach to Diophantine Equations, IHP notes.

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Recent:

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(2) J.-L. Verger-Gaugry, A Dobrowolski-type inequality for the poles of the dynamical zeta function of the beta-shift, submitted (2022).
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(3) J.-L. Verger-Gaugry, An universal minoration of the Mahler measure of real reciprocal algebraic integers, (2022).
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(4) J.-L. Verger-Gaugry, A Panorama on the Minoration of the Mahler Measure: from the Problem of Lehmer to its Reformulations in Topology and Geometry,
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