On the solutions of some generalized Lebesgue-Ramanujan-Nagell equations

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(This is joint work with Elif Kızıldere Mutlu and Maohua Le)

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- Six years later, V.A. Lebesgue solved (1) for $m \equiv 0 \pmod 2$, namely, he proved that the equation

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1.a. Motivation

1.a.1. History

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has no solutions (x, y, n).

- Afterwards, T. Nagell solved the equation with the form

$$x^2 + D = y^n,\ x, y, n \in \mathbb{Z}^+,\ \gcd(x, y) = 1,\ n > 2, \quad (2)$$

where $D = 3, 4$ and 5. Therefore, (2) and its generalizations are called the generalized **Lebesgue-Nagell** equation.

- In 1913, S. Ramanujan conjectured that the equation

$$x^2 + 7 = 2^{n+2},\ x, n \in \mathbb{Z}^+ \tag{3}$$

has only the solutions $(x, n) = (1, 1), (3, 2), (5, 3), (11, 5)$ and $(181, 13)$.

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- In 1945, W. Ljunggren proposed the same problem in Norwegian and T. Nagell first proved it three years later. Therefore, the equation

$$x^2 + D = \begin{cases} 2^{n+2}, & \text{if } p = 2, \\ & \qquad\qquad x, n \in \mathbb{Z}^+ \\ p^n, & \text{if } p \neq 2, \end{cases} \tag{4}$$

and its generalizations are called **the generalized Ramanujan-Nagell equation**.

- Some details about the generalized Lebesgue-Ramanujan-Nagell equations can be found in the following survey papers:
    1. F.S. Abu Muriefah and Y. Bugeaud, 2006, Rev. Colombiana Math.
    2. A. Bérczes and I. Pink, 2014, An. St. Univ. Ovidius Constanta.
    3. M.H. Le and G.Soydan, 2020, Survey in Mathematics and its Applications.

In 2014, N. Terai proposed the following conjecture:

Conjecture 1

For any k with $k > 1$, the equation

$$x^2 + (2k-1)^y = k^z, \;\; x, y, z \in \mathbb{Z}^+ \tag{5}$$

has only one solution $(x, y, z) = (k-1, 1, 2)$.

- (2014) For the case $4 \mid k$, N. Terai used some classical methods to discuss the eq. $x^2 + (2k-1)^y = k^z$ for $k \leq 30$. However, his method does not apply for $k \in \{12, 24\}$.

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- (2020) Y. Fujita and N. Terai verified the conjecture when the cases $2k-1 = 3p^\ell$ and $2k-1 = 5p^\ell$ with p a prime and ℓ a positive integer.

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- (2020) Y. Fujita and N. Terai verified the conjecture when the cases $2k-1 = 3p^\ell$ and $2k-1 = 5p^\ell$ with p a prime and ℓ a positive integer.
- Most of solved cases of Conjecture 1 focus on the case $4 \nmid k$, and very little is known in the case $4 \mid k$.

In this work, using the modular approach we prove the following result:

Theorem 1 (Mutlu-Le-S, 2022)

If $4 \mid k$, $30 < k < 724$ *and* $2k - 1$ *is an odd prime power, then under the GRH, Conjecture 1 is true.*

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- The method of using such results to deal with Diophantine problems, is called the ***modular approach***. After Wiles' proof, the original strategy was strengthened and many mathematicians achieved great success in solving other equations that previously seemed hard.
- As a result of these efforts, the generalized Fermat equation

$$Ax^p + By^q = Cz^r, \text{ with } 1/p + 1/q + 1/r < 1, \tag{6}$$

where $p, q, r \in \mathbb{Z}_{\geq 2}$, A, B, C are non-zero integers and x, y, z are unknown integers became a new area of interest.

- Call an integer solution (x, y, z) to such an equation *proper* if $\gcd(x, y, z) = 1$. It was proved that equation

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- The following survey papers are good references for the case $ABC = 1$:
  1. M.A. Bennett, I. Chen, S. Dahmen and S. Yazdani, Generalized Fermat equations : a miscellany, Int. J. Number Theory, (2015).
  2. M. A. Bennett, P. Mihăilescu and S. Siksek, The generalized Fermat equation, in Springer volume Open Problems in Mathematics, (2016).

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- One can find the details concerning modular approach in Cohen's book (Chapter 15) and the paper is titled "Modular Approach to Diophantine equations" of Samir Siksek (2012).

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- Here we give recipes for signature $(n, n, 2)$ which was firstly described by M.A. Bennett and C. Skinner.
- We denote by $rad(m)$ the radical of $|m|$, i.e. the product of distinct primes dividing m, and by $\text{ord}_p(m)$ the largest nonnegative integer k such that p^k divides m.

- We always assume that $n \geq 7$ is prime, and a, b, c, A, B and C are nonzero integers with Aa, Bb and Cc pairwise coprime, A and B are nth-power free, C squarefree satisfying

$$Aa^n + Bb^n = Cc^2. \tag{7}$$

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- We further assume that we are in one of the following situations:
  (i) $abABC \equiv 1 \pmod 2$ and $b \equiv -BC \pmod 4$.
  (ii) $ab \equiv 1 \pmod 2$ and either $\mathrm{ord}_2(B) = 1$ or $\mathrm{ord}_2(C) = 1$.
  (iii) $ab \equiv 1 \pmod 2$, $ord_2(B) = 2$ and $C \equiv -bB/4 \pmod 4$.
  (iv) $ab \equiv 1 \pmod 2$, $ord_2(B) \in \{3, 4, 5\}$ and $c \equiv C \pmod 4$.
  (v) $\mathrm{ord}_2(Bb^n) \geq 6$ and $c \equiv C \pmod 4$.

For our eq. we are in case (v) and we will consider the curve

$$E_3(a,b,c): Y^2 + XY = X^3 + \frac{cC-1}{4}X^2 + \frac{BCb^n}{64}X, \tag{8}$$

which is defined over $\mathbb{Q}$. By a lemma of Bennett-Skinner, the conductor of the curve $E = E_3(a,b,c)$ is given by

$$N(E) = 2^{\alpha} \cdot C^2 \cdot rad(abAB) \tag{9}$$

where

$$\alpha = \begin{cases} -1 & \text{if } i = 3, \text{ case } (v) \text{ and } \operatorname{ord}_2(Bb^n) = 6, \\ 0 & \text{if } i = 3, \text{ case } (v) \text{ and } \operatorname{ord}_2(Bb^n) \geq 7. \end{cases}$$

Here and below, we assume that (x, y, z) is a solution of the equation

$$x^2 + (2k-1)^y = k^z, \; x, y, z \in \mathbb{Z}^+ \tag{10}$$

with $(x, y, z) \neq (k-1, 1, 2)$. Then, by (10), we can get $z > y$ immediately. Obviously, if we can prove that the solution (x, y, z) does not exist, then Conjecture 1 is true. The following two lemmas are basic properties on the solution (x, y, z).

Lemma 2 (Mutlu-Le-S, 202)

Suppose $4 \mid k$. *Then* $2 \nmid y$.

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Proof.

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Lemma 3 (Mutlu-Le-S, 2022)

If $2k - 1$ *is an odd prime power, then* $2 \nmid z$.

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If $2k - 1$ *is an odd prime power, then* $2 \nmid z$.

Proof.

See Lemma 2.6 of Deng, Guo and Xu. □

Lemma 4

Let $F(t) = t + a/t$ be a function of the real variable t, where a is a constant with $a > 1$. Then $F(t)$ is a strictly decreasing function for $1 \leq t < \sqrt{a}$.

Proof.

Since $F'(t) = 1 - a/t^2 < 0$ for $1 \leq t < \sqrt{a}$, where $F'(t)$ is the derivative of $F(t)$, we obtain the lemma immediately. □

Lemma 5 (Mutlu-Le-S, 2022)

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Lemma 6 (Mutlu-Le-S, 2022)

If $4 \mid k$, $2k - 1$ is an odd prime power and k is a square, then Conjecture 1 is true.

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Lemma 6 (Mutlu-Le-S, 2022)

If $4 \mid k$, $2k - 1$ is an odd prime power and k is a square, then Conjecture 1 is true.

Proof.

The proof is elementary and it is based on Lemma 2. □

For any fixed positive integers m and n with $n > 1$, there exist unique positive integers f and g such that

$$m = fg^n, \ f \text{ is } n\text{th-power free.} \qquad (11)$$

The positive integer f is called the nth-power free part of m, and denoted by $f(m)$. Similarly, g is denoted by $g(m)$. Obviously, by (11), if $2 \mid m$, then we have

$$\mathrm{ord}_2(m) = \mathrm{ord}_2(f(m)) + n\,\mathrm{ord}_2(g(m)), \ 0 \leq \mathrm{ord}_2(f(m)) < n. \qquad (12)$$

Let k be a positive integer with $4 \mid k$. Suppose that $y \geq 7$ is prime. Then equation $x^2 + (2k-1)^y = k^z$ becomes

$$(-1) \cdot (2k-1)^y + k^z = x^2. \tag{13}$$

Then, the ternary equation $Ax^p + By^q = Cz^r$ can be obtained from (13) by the substitution

$$A = -1,\ a = 2k-1,\ B = f(k^z),\ b = g(k^z),\ C = 1,\ c = (-1)^{(x-1)/2}x, \tag{14}$$

where $f(k^z)$ and $g(k^z)$ are defined as in (11).

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where $f(k^z)$ and $g(k^z)$ are defined as in (11).

Lemma 7 (Mutlu-Le-S, 202?)

If $4 \mid k$, then b, B and C in (14) satisfy the case (v) [$\mathrm{ord}_2(Bb^n) \geq 6$ and $c \equiv C \pmod 4$] with $\mathrm{ord}_2(Bb^n) > 6$.

- By the strategy of Bennett-Skinner, we are interested in the following elliptic curve (called a Frey curve)

$$E_3 : Y^2 + XY = X^3 + \frac{(-1)^{(x-1)/2}x - 1}{4}X^2 + \frac{k^z}{64}X. \qquad (15)$$

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- Note that when $k = 720$, one gets that $N(E_3) = 43170$ and $2k - 1 = 1439$ is prime. But when $k = 724$, one obtains $N(E_3) = 523.814$, outside the range of the Cremona elliptic curve database where the upper bound for conductors is 500.000 (in November 2021).

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- We therefore restrict attention to $30 < k \leq 720$.

- Using Lemmas 5 and 6, we can exclude the cases $k = 2^{r_0}$ with $r_0 = 6$ and $k = \ell^2$ with $\ell \in \{6, 8, 10, 18, 22, 24\}$. This leaves 50 values of k to consider. We proceed as follows.

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- For a given k we compute by the formula the conductor of the Frey curve

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  Using Cremona's elliptic curve database we obtain a list of isomorphism classes of elliptic curves for that conductor.
- In each class, we must determine whether there exists a model consistent with the model at (17).

- For example, when $k = 192$ and the conductor is 2298, the isomorphism class of the curve labelled 2298.h4, $[1, 0, 0, -184, 1088]$, contains the curve $[1, -48, 0, 576, 0]$ (note that this fails to provide a solution to our problem, because the corresponding values of y, z, equal 1,2, not allowed).

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- Since our Frey curve has point $(0, 0)$ of order 2, it is only necessary to consider isomorphism classes determined by curves with nontrivial 2-torsion.

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- Since our Frey curve has point $(0, 0)$ of order 2, it is only necessary to consider isomorphism classes determined by curves with nontrivial 2-torsion.
- Suppose a Cremona class representative has nontrivial 2-torsion point T_0. To obtain an isomorphic curve of the form (15) we must take the transformation mapping T_0 to $(0, 0)$, and then test the resulting curve to see whether the $X-$ coefficient is of the form $\frac{k^z}{64}$. This was programmed into Magma.

- Resulting curves with corresponding $(y, z) = (1, 2)$ are not allowed, and only one other curve resulted, namely $[1, 733/4, 0, 33/16, 0]$ when $k = 132$ with $z = 2$. But this does not provide a solution to our problem because there is no corresponding value of x (or y). Finally, thus, we deduce that the eq. $x^2 + (2k-1)^y = k^z$ has no solutions where $y \geq 7$ and $30 < k \leq 720$.

- Here we solve the Diophantine equations

$$x^2 + (2k-1)^3 = k^z, \ z > 3 \text{ odd}, \tag{18}$$

and

$$x^2 + (2k-1)^5 = k^z, \ z > 5 \text{ odd}, \tag{19}$$

where $4 \mid k$, $30 < k < 724$ and $2k - 1$ is an odd prime power.

- Write in the eq. $x^2 + (2k-1)^y = k^z$, $y = 6A + i$, $z = 3B + j$ where $i = 3$ or 5 and $0 \leq j \leq 2$, $A, B \geq 0$. Since $2k-1$ is an odd prime power, we have $2k - 1 = p^r$, where p is an odd prime and r is a positive integer. Then we see that

$$\left(\frac{k^{B+j}}{(2k-1)^{2A}}, \frac{xk^j}{(2k-1)^{3A}} \right)$$

is an S-integral point (U, V) on the elliptic curve

$$\mathcal{E}_{ijk} : V^2 = U^3 - (2k-1)^i k^{2j} ,$$

where $S = \{p\}$, $4 \mid k$, $30 < k < 724$ and $2k - 1$ is a power of p, in view of the restriction $\gcd(k, x) = 1$.

- A practical method for the explicit computation of all S-integral points on a Weierstrass elliptic curve has been developed by A. Pethő, H.G. Zimmer, J. Gebel and E. Herrmann and has been implemented in Magma. The relevant routine `SIntegralPoints` worked without problems for all triples (i, j, k) except for

$$(i, j, k) \in \{(5, 2, 96), (5, 1, 120), (5, 2, 156), (5, 2, 180), (5, 2, 192),$$

$$(5, 2, 220), (3, 1, 232), (5, 0, 232), (5, 2, 232), (5, 0, 240), (5, 2, 240),$$

$$(5, 2, 244), (5, 0, 304), (5, 1, 304), (5, 2, 304), (3, 2, 316), (5, 0, 316),$$

$$(5, 2, 316), (5, 2, 324), (5, 0, 360), (5, 1, 364), (5, 2, 364), (3, 2, 372),$$

$$(5, 1, 372), (5, 2, 372), (5, 2, 376), (3, 1, 412), (3, 2, 412), (5, 0, 412),$$

$$(5, 0, 420), (5, 0, 432), (5, 1, 432), (3, 2, 444), (5, 1, 444), (5, 2, 444),$$

$$(5, 0, 456), (5, 1, 456), (5, 2, 460), (5, 1, 492), (5, 1, 516), (5, 2, 516),$$

- $$(3,1,520),(5,0,520),(5,2,520),(5,2,532),(5,1,544),(5,2,552),$$
  $$(3,2,612),(5,0,612),(5,1,612),(5,2,612),(5,1,616),(5,0,640),$$
  $$(5,2,640),(3,2,652),(5,2,652),(5,2,660),(3,2,664)(5,0,664),$$
  $$(5,2,664),(5,1,684),(5,0,700),(5,1,700),(5,2,700),(5,0,712),$$
  $$(5,0,720),(5,1,720)\}$$
- The non-exceptional triples (i,j,k) do not give any positive integer solution to equation $x^2+(2k-1)^3=k^z$ or $x^2+(2k-1)^5=k^z$.

Thirty-eight of the above exceptional triples have been solved using an elementary approach, as follows.

Lemma 8 (Mutlu-Le-S, 202?)

If $k \equiv 3$ *or* $4 \pmod 5$, *then the equation*

$$x^2 + (2k-1)^5 = k^z, \ z > 5 \text{ odd},$$

has no solutions (x, z), *where* $4 \mid k$, $30 < k < 724$ *and* $2k-1$ *is an odd prime power.*

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Lemma 8 (Mutlu-Le-S, 202?)

If $k \equiv 3$ or $4 \pmod 5$, then the equation

$$x^2 + (2k-1)^5 = k^z, \ z > 5 \text{ odd},$$

has no solutions (x, z), where $4 \mid k$, $30 < k < 724$ and $2k-1$ is an odd prime power.

Proof.

The proof is based on congruences and Jacobi symbol. □

Lemma 9 (Mutlu-Le-S, 202?)

If $2 \mid k$ *and* $k+1$ *has an odd prime divisor* p *with* $p \equiv \pm 3 \pmod 8$*, then the eq.*

$$x^2 + (2k-1)^5 = k^z, \ z > 5 \text{ odd},$$

has no solutions (x, z)*, where* $4 \mid k$*,* $30 < k < 724$ *and* $2k-1$ *is an odd prime power.*

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The proof is based on congruences and Jacobi symbol. □

Notice that if $2 \mid k$ and every odd prime divisor p of $k+1$ satisfies $p \equiv \pm 1 \pmod 8$, then either $k+1 \equiv 1 \pmod 8$ or $k+1 \equiv -1 \pmod 8$. Hence, by Lemma 9, we can obtain the following lemma immediately.

Lemma 10 (Mutlu-Le-S, 202?)

If $k \equiv 2$ or $4 \pmod 8$, then the eq. $x^2 + (2k-1)^5 = k^z$ has no solutions (x, z).

Notice that if $2 \mid k$ and every odd prime divisor p of $k+1$ satisfies $p \equiv \pm 1 \pmod 8$, then either $k+1 \equiv 1 \pmod 8$ or $k+1 \equiv -1 \pmod 8$. Hence, by Lemma 9, we can obtain the following lemma immediately.

Lemma 10 (Mutlu-Le-S, 202?)

If $k \equiv 2$ or $4 \pmod 8$, then the eq. $x^2 + (2k-1)^5 = k^z$ has no solutions (x, z).

Lemma 11 (Mutlu-Le-S, 202?)

For $k \in \{120, 156, 180, 220, 244, 304, 316, 324, 360, 364, 372, 376, 412, 420, 444, 460, 492, 516, 532, 544, 612, 652, 660, 664, 684, 700\}$, the eq. $x^2 + (2k-1)^5 = k^z$ has no solutions (x, z).

Proof.

By Lemmas 8, 9 and 10, the eq. $x^2 + (2k-1)^5 = k^z$ has no solutions (x, z) for $k \in \{244, 304, 324, 364, 444, 544, 664, 684\}$, $k \in \{120, 360, 376\}$ and $k \in \{156, 180, 220, 316, 372, 412, 420, 460, 492, 516, 532, 612, 652, 660, 700\}$, respectively. □

- Denote the rank of the elliptic curve $\mathcal{E}_{ijk}$ by r. Here, we seperate the above remaining **twentynine exceptional triples** (i, j, k) depending on whether $r = 0$, $r = 1$ and $r = 2$, respectively.

- Denote the rank of the elliptic curve $\mathcal{E}_{ijk}$ by r. Here, we seperate the above remaining **twentynine exceptional triples** (i, j, k) depending on whether $r = 0$, $r = 1$ and $r = 2$, respectively.

- For the triples $(i, j, k) \in \{(5, 1, 456), (5, 2, 552), (5, 1, 616),$
(3,2,652),(5,1,720)},
there are no rational points (so no S-integral points) on $\mathcal{E}_{ijk}$ under the assumption that $r = 0$ which is proved by descent algorithms of Magma.

- For each remaining triples (i, j, k) (in total **twentyfour triples**), the rank of $\mathcal{E}_{ijk}$ is 1, i.e. $r = 1$ except for $(i, j, k) = (3, 2, 664)$. We performed two-, four- and eight-descent algorithms or two-, four-, three- and twelve-descent algorithms for these triples (These algorithms were improved by J. Cremona, S. Donnelly, T. Fisher, J. Merriman, C. O'Neil, S. Siksek, D. Simon, N. Smart, S. Stamminger, M. Stoll, $\cdots$).

- For each remaining triples (i, j, k) (in total **twentyfour triples**), the rank of $\mathcal{E}_{ijk}$ is 1, i.e. $r = 1$ except for $(i, j, k) = (3, 2, 664)$. We performed two-, four- and eight-descent algorithms or two-, four-, three- and twelve-descent algorithms for these triples (These algorithms were improved by J. Cremona, S. Donnelly, T. Fisher, J. Merriman, C. O'Neil, S. Siksek, D. Simon, N. Smart, S. Stamminger, M. Stoll, $\cdots$).
- Since Magma found a generator for each of them, it was succesfull to show non-existence of S-integral points on $\mathcal{E}_{ijk}$ for the exceptional twentythree triples.

- The rank 1 curves frequently have generators of large height. We can estimate the height of a generator in advance using the Gross-Zagier formula, as for example used by A. Bremner in treating the family of curves $y^2 = x(x^2 + p)$.

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- Having an estimate of the height in advance tells us whether it is likely that standard descent arguments, as programmed into Magma, will be successfull in finding the generator.

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- Having an estimate of the height in advance tells us whether it is likely that standard descent arguments, as programmed into Magma, will be successfull in finding the generator.
- For **twentythree curves** we are considering here, Magma was able to compute generators for all cases, using a combination of three-, four-, eight-, and twelve-descent algorithms.

- The single instance of rank 2 was at $(i, j, k) = (3, 2, 664)$. Magma found only S-integral point $(6435758912 : 516297057335360 : 1)$ on the corresponding curve under the assumption that $1 \leq r \leq 2$ and its generator $(402234932 : 8067141520865 : 1)$.

- The single instance of rank 2 was at $(i, j, k) = (3, 2, 664)$. Magma found only S-integral point $(6435758912 : 516297057335360 : 1)$ on the corresponding curve under the assumption that $1 \leq r \leq 2$ and its generator $(402234932 : 8067141520865 : 1)$.
- By eight-descent algorithm, we found two independent points which are generators. So, it is confirmed that $r = 2$ and this curve has only S-integral point $(6435758912 : 516297057335360 : 1)$. But it does not give any positive integer solution to equation $x^2 + (2k-1)^3 = k^z$.
- To sum up, the main result is proved.

Let k be a fixed positive integer with $k > 1$. Here, using various elementary methods in number theory, we give certain criterions which can make the equation

$$x^2 + (2k-1)^y = k^z \tag{20}$$

to have no positive integer solutions (x, y, z) with $y \in \{3, 5\}$. These results make up the defiency of the modular approach when applied to the above equation.

Theorem 12 (Mutlu-Le-S, 202?)

If $(2k-1)$ *has a divisor* d *with* $d \equiv \pm 3 \pmod 8$, *then the eq.* $x^2 + (2k-1)^y = k^z$ *has no solutions* (x, y, z) *with* $y \in \{3, 5\}$.

For any sufficiently large positive integer N, let N_0 denote the number of positive integers k which satisfy the assumption of Theorem 12 with $1 \leq k \leq N$. Then we have

$$\lim_{N \to \infty} \frac{N_0}{N} \sim 1 - \prod_{p} (1 - \frac{1}{p}) \text{ (p is prime with } p \equiv \pm 3 \pmod 8)$$

It implies that there exist almost all positive integers k can make $x^2 + (2k-1)^y = k^z$ has no solutions (x, y, z) with $y \in \{3, 5\}$.

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It implies that there exist almost all positive integers k can make $x^2 + (2k-1)^y = k^z$ has no solutions (x, y, z) with $y \in \{3, 5\}$.

Theorem 13 (Mutlu-Le-S, 202?)

If k is a square, then the eq. $x^2 + (2k-1)^y = k^z$ has no solutions (x, y, z) with $y \in \{3, 5\}$.

Theorem 14 (Mutlu-Le-S, 202?)

If k is not a square, and (x, y, z) is a solution of the eq. $x^2 + (2k-1)^y = k^z$ with $y \in \{3, 5\}$, then $2 \nmid z$ and

$$y = Z_1 t, \ t \in \mathbb{Z}^+,$$

$$x + k^{(z-1)/2}\sqrt{k} = (X_1 + \lambda Y_1\sqrt{k})^t(u + v\sqrt{k}), \ \lambda \in \{1, -1\},$$

where X_1, Y_1, Z_1 are positive integers such that

$$X_1^2 - kY_1^2 = (-(2k-1))^{Z_1}, \ \gcd(X_1, Y_1) = 1, \ Z_1 \mid h(4k)$$

and

$$1 < \left|\frac{X_1 + Y_1\sqrt{k}}{X_1 - Y_1\sqrt{k}}\right| < u_1 + v_1\sqrt{k},$$

where $h(4k)$ is the class number of binary quadratic primitive forms with discriminant $4k$, (u, v) is a solution of Pell's equation

$$u^2 - kv^2 = 1, \ u, v \in \mathbb{Z}, \tag{21}$$

and (u_1, v_1) is the least solution of (21).

Let $\mathbb{Z}[t]$ denote the set of all the polynomials of indeterminate t with integer coefficients. It is a well known fact that if $F(t) \in \mathbb{Z}[t]$ which leading coefficient is positive, then there exist positive integers m which can make

$$F(t) \in \mathbb{Z}^{+},\ t \in \mathbb{Z}^{+},\ t \geq m. \tag{22}$$

Therefore, we may use the notation $m(F(t))$ to represent the least value of positive integers m with (22).

Lemma 15

Let $F(t) = t + a/t$ be a function of the real variable t, where a is a constant with $a > 1$. Then $F(t)$ is a strictly decreasing function for $1 \leq t < \sqrt{a}$.

Proof.

Since $F'(t) = 1 - a/t^2 < 0$ for $1 \leq t < \sqrt{a}$, where $F'(t)$ is the derivative of $F(t)$, we obtain the lemma immediately. □

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Lemma 16 (Mutlu-Le-S, 202?)

Let $F(t) = t^{2n} - a_{2n-1}t^{2n-1} - \cdots - a_0 \in \mathbb{Z}[t]$, *where* n *is a positive integer. If there exist* $G(t), R(t) \in \mathbb{Z}[t]$ *such that*

$$F(t) = (G(t))^2 + R(t), \tag{23}$$

where

$$G(t) = t^n - b_{n-1}t^{n-1} - \cdots - b_0,\ R(t) = r_\ell t^\ell - r_{\ell-1}t^{\ell-1} - \cdots - r_0,\ r_\ell \neq 0,\ \ell < n, \tag{24}$$

cont.

then the equation

$$X^2 = F(Y), \; X, Y \in \mathbb{Z}^+ \tag{25}$$

has no solutions (X, Y) with $Y \geq Y_0$, where

$$Y_0 = \begin{cases} \max\{m(G(t)), \; m(R(t)), \; m(2G(t) - R(t))\}, & \text{if } r_\ell > 0, \\ \max\{m(G(t)), \; m(-R(t)), \; m(2G(t) + R(t) - 1)\}, & \text{if } r_\ell < 0. \end{cases} \tag{26}$$

Lemma 17 (Mutlu-Le-S, 202?)

Each of the following equations has no solutions (X, Y).

$$X^2 = Y^4 - 8Y^3 + 12Y^2 - 6Y + 1, \quad X, Y \in \mathbb{Z}^+. \tag{27}$$

$$X^2 = Y^6 - 8Y^3 + 12Y^2 - 6Y + 1, \quad X, Y \in \mathbb{Z}^+. \tag{28}$$

$$X^2 = Y^6 - 32Y^5 + 80Y^4 - 80Y^3 + 40Y^2 - 10Y + 1, \quad X, Y \in \mathbb{Z}^+. \tag{29}$$

$$X^2 = Y^8 - 32Y^5 + 80Y^4 - 80Y^3 + 40Y^2 - 10Y + 1, \quad X, Y \in \mathbb{Z}^+. \tag{30}$$

$$X^2 = Y^{10} - 8Y^6 + 12Y^4 - 6Y^2 + 1, \quad X, Y \in \mathbb{Z}^+. \tag{31}$$

cont.

$$X^2 = Y^{10} - 32Y^5 + 80Y^4 - 80Y^3 + 40Y^2 - 10Y + 1, \quad X, Y \in \mathbb{Z}^+. \quad (32)$$

$$X^2 = Y^{14} - 32Y^{10} + 80Y^8 - 80Y^6 + 40Y^4 - 10Y^2 + 1, \quad X, Y \in \mathbb{Z}^+. \quad (33)$$

$$X^2 = Y^{18} - 32Y^{10} + 80Y^8 - 80Y^6 + 40Y^4 - 10Y^2 + 1, \quad X, Y \in \mathbb{Z}^+. \quad (34)$$

cont.

$$X^2 = Y^{10} - 32Y^5 + 80Y^4 - 80Y^3 + 40Y^2 - 10Y + 1, \quad X, Y \in \mathbb{Z}^+. \quad (32)$$

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$$X^2 = Y^{18} - 32Y^{10} + 80Y^8 - 80Y^6 + 40Y^4 - 10Y^2 + 1, \quad X, Y \in \mathbb{Z}^+. \quad (34)$$

Proof.

The proof is based on Lemma 16. □

Lemma 18 (Mutlu-Le-S, 202?)

If (x, y, z) *is a solution of the eq.* $x^2 + (2k-1)^y = k^z$ *with* $y \in \{3, 5\}$*, then* $2 \nmid z$*.*

Lemma 18 (Mutlu-Le-S, 202?)

If (x, y, z) is a solution of the eq. $x^2 + (2k-1)^y = k^z$ with $y \in \{3, 5\}$, then $2 \nmid z$.

Proof.

The proof is based on Lemmas 15 and 17 □

- Let D be a fixed nonsquare positive integer, and let $h(4D)$ denote the class number of binary quadratic primitive forms with discriminant $4D$. Further let K be a fixed odd integer with $|K| > 1$ and $\gcd(D, K) = 1$. It is well known that Pell's equation

$$U^2 - DV^2 = 1, \;\; U, V \in \mathbb{Z} \tag{35}$$

has positive integer solutions (U, V), and it has a unique positive integer solution (U_1, V_1) such that $U_1 + V_1\sqrt{D} \leq U + V\sqrt{D}$, where (U, V) through all positive integer solutions of (35).

- Let D be a fixed nonsquare positive integer, and let $h(4D)$ denote the class number of binary quadratic primitive forms with discriminant $4D$. Further let K be a fixed odd integer with $|K| > 1$ and $\gcd(D, K) = 1$. It is well known that Pell's equation

$$U^2 - DV^2 = 1, \ U, V \in \mathbb{Z} \tag{35}$$

has positive integer solutions (U, V), and it has a unique positive integer solution (U_1, V_1) such that $U_1 + V_1\sqrt{D} \leq U + V\sqrt{D}$, where (U, V) through all positive integer solutions of (35).
- The solution (U_1, V_1) is called the least solution of (35). For any positive integer n, let

$$U_n + V_n\sqrt{D} = (U_1 + V_1\sqrt{D})^n.$$

Then $(U, V) = (U_n, V_n)$ $(n = 1, 2, \cdots)$ are all positive integer solutions of (35).

It follows that every solution (U, V) of $U^2 - DV^2 = 1$ can be expressed as

$$U+V\sqrt{D} = \lambda_1(U_1+\lambda_2 V_1\sqrt{D})^m,\ \lambda_1, \lambda_2 \in \{1, -1\},\ m \in \mathbb{Z},\ m \geq 0. \quad (36)$$

Hence, by (36), every solution (U, V) of (35) satisfies

$$V \equiv 0 \pmod{V_1}. \quad (37)$$

Lemma 19 (Le-1995, Yang and Fu-2015)

If the equation

$$X^2 - DY^2 = K^Z,\ X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0 \quad (38)$$

has solutions (X, Y, Z), then every solution (X, Y, Z) of (38) can be expressed as

$$Z = Z_1 t,\ t \in \mathbb{Z}^+,$$

$$X + Y\sqrt{D} = (X_1 + \lambda Y_1 \sqrt{D})^t (U + V\sqrt{D}),\ \lambda \in \{1, -1\},$$

where (U, V) is a solution of (35) and X_1, Y_1, Z_1 are positive integers satisfy

$$X_1^2 - DY_1^2 = K^{Z_1},\ \gcd(X_1, Y_1) = 1,\ Z_1 \mid h(4D) \tag{39}$$

and

$$1 < \left| \frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}} \right| < U_1 + V_1\sqrt{D}, \tag{40}$$

and (U_1, V_1) is the least solution of (35).

Proof of Theorem 12.

We now assume that (x, y, z) is a solution of the eq. $x^2 + (2k-1)^y = k^z$ with $y \in \{3, 5\}$. By Lemma 18, we have $2 \nmid z$. Hence, for any divisor d of $2k-1$, we get from (5) that

$$1 = \left(\frac{k^z}{d}\right) = \left(\frac{k}{d}\right) = \left(\frac{4k}{d}\right) = \left(\frac{2}{d}\right), \tag{41}$$

where $(*/*)$ is the Jacobi symbol. However, if $d \equiv \pm 3 \pmod 8$, then we have $(2/d) = -1$, which contradicts (41). Thus, the theorem is proved. □

Proof of Theorem 13.

Since k is square, we have

$$k = \ell^2, \ \ell \in \mathbb{N}. \tag{42}$$

Substitute (42) into $x^2 + (2k-1)^3 = k^z$ and $x^2 + (2k-1)^5 = k^z$, we get

$$x^2 + (2\ell^2 - 1)^3 = \ell^{2z}, \ x, z \in \mathbb{N}, \ z > 3 \tag{43}$$

and

$$x^2 + (2\ell^2 - 1)^5 = \ell^{2z}, \ x, z \in \mathbb{N}, \ z > 5, \tag{44}$$

respectively. Using Lemmas 15, 17 and 18, we can complete the proof. □

Proof of Theorem 14.

Notice that k is not a square, $2 \nmid 2k-1$, $\gcd(k, 2k-1) = 1$ and $2 \nmid z$ by Lemma 18, then the equation

$$X^2 - kY^2 = (-(2k-1))^Z, \ X, Y, Z \in \mathbb{Z}, \ \gcd(X, Y) = 1, \ Z > 0 \quad (45)$$

has a solution

$$(X, Y, Z) = (x, k^{(z-1)/2}, y). \quad (46)$$

Therefore, apply Lemma 19 to (45) and (46), we can obtain the theorem immediately. □

- Now we illustrate Theorem 14 for determining whether the eq. $x^2 + (2k-1)^5 = k^z$ has solutions with $y \in \{3, 5\}$ for $k = 736$. These two cases correspond to the Diophantine equations

$$x^2 + 1471^3 = 736^z, \; x, z \in \mathbb{N}, \tag{47}$$

$$x^2 + 1471^5 = 736^z, \; x, z \in \mathbb{N}, \tag{48}$$

respectively. Here we will only solve the equation (47). By the same techniques, we can similarly treat the latter one.

- Now we illustrate Theorem 14 for determining whether the eq. $x^2 + (2k-1)^5 = k^z$ has solutions with $y \in \{3,5\}$ for $k = 736$. These two cases correspond to the Diophantine equations

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$$x^2 + 1471^5 = 736^z, \ x, z \in \mathbb{N}, \tag{48}$$

respectively. Here we will only solve the equation (47). By the same techniques, we can similarly treat the latter one.

- We now to prove that (47) has no solutions (x, z). To do this, we use Lemma 19.

- If (x, z) is a solution of (47), then the equation

$$X^2 - 736Y^2 = (-1471)^Z, \; X, Y, Z \in \mathbb{Z}, \; \gcd(X, Y) = 1, \; Z > 0 \quad (49)$$

has a solution

$$(X, Y, Z) = (x, 736^{(z-1)/2}, 3). \quad (50)$$

- If (x, z) is a solution of (47), then the equation

$$X^2 - 736Y^2 = (-1471)^Z,\ X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0 \quad (49)$$

has a solution

$$(X, Y, Z) = (x, 736^{(z-1)/2}, 3). \quad (50)$$

- Let (X_1, Y_1, Z_1) be a solution of (49). Applying Lemma 19 to (50), we have

$$3 = Z_1 t,\ t \in \mathbb{Z}^+. \quad (51)$$

On the other hand, since $h(4 \times 736) = 4$, by (39), we have $4 \equiv 0 \pmod{Z_1}$.

- Hence, we see from (51) that $Z_1 = 1$. So we have

$$X_1^2 - 736Y_1^2 = -1471,\ X_1, Y_1 \in \mathbb{Z}^+,\ \gcd(X_1, Y_1) = 1. \tag{52}$$

Further, since the least solution of Pell's equation

$$U^2 - 736V^2 = 1,\ U, V \in \mathbb{Z}$$

is $(U_1, V_1) = (24335, 897)$, by (40), we have

$$1 < \left| \frac{X_1 + Y_1\sqrt{736}}{X_1 - Y_1\sqrt{736}} \right| < 24335 + 897\sqrt{736}. \tag{53}$$

- Hence, we see from (51) that $Z_1 = 1$. So we have

$$X_1^2 - 736Y_1^2 = -1471,\ X_1, Y_1 \in \mathbb{Z}^+,\ \gcd(X_1, Y_1) = 1. \tag{52}$$

Further, since the least solution of Pell's equation

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$$1 < \left| \frac{X_1 + Y_1\sqrt{736}}{X_1 - Y_1\sqrt{736}} \right| < 24335 + 897\sqrt{736}. \tag{53}$$

- Hence, by (52) and (53), we get

$$X_1 + Y_1\sqrt{736} < \sqrt{1471(24335 + 897\sqrt{736})} < 8462. \tag{54}$$

- Using MAPLE, by Lemma 19, we see that the only solution (X_1, Y_1, Z_1) of (52) is $(2577, 95, 1)$. Then we have

$$x+736^{(z-1)/2}\sqrt{736}=(2577+95\lambda\sqrt{736})^3(U+V\sqrt{736}),\ \lambda\in\{1,-1\}, \tag{55}$$

where (U, V) is a solution of Pell's equation

$$U^2-736V^2=1,\ U,V\in\mathbb{Z}. \tag{56}$$

Let

$$f+g\lambda\sqrt{736}=(2577+95\lambda\sqrt{736})^3. \tag{57}$$

- Obviously, f and g are positive integers. Substitute (57) into (55), we have

$$x+736^{(z-1)/2}\sqrt{736}=(f+g\lambda\sqrt{736})(U+V\sqrt{736}),$$

whence we get

$$736^{(z-1)/2} = fV + \lambda gU. \tag{58}$$

- Since the least solution of (56) is $(U_1, V_1) = (24335, 897)$, by (37), we have $V \equiv 0 \pmod{897}$. So, we obtain $23 \mid V$. Hence, by (58), we get

$$0 \equiv \lambda gU \pmod{23}. \tag{59}$$

Further, since $\lambda \in \{1, -1\}$ and $\gcd(U, 23) = 1$ by (57), we see from (59) that

$$g \equiv 0 \pmod{23}. \tag{60}$$

- However, by (57), we have $g = 2523692765 \equiv 9 \pmod{23}$. It implies that (60) is false. Therefore, (47) has no solutions (x, z).

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Thank you for your attention!

Merci pour votre attention!