On Some Diophantine equations with power sums

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Workshop on Diophantine Equations and Algebraic Equations, May 2-3, 2023
Department of Mathematics, Savoie Mont Blanc University
Chambery-FRANCE

(This is joint work with Daniele Bartoli and Maohua Le)

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1. About Bernoulli numbers and Bernoulli polynomials

- In his posthumous book "Ars Conjectandi" published in 1713 Swiss mathematician **Jakob Bernoulli** (1655-1705)

introduced the Bernoulli numbers in connection to the study of the sums of powers of consecutive integers $1^k + 2^k + \cdots + n^k$. After listing the formulas for the sums of powers:

1. About Bernoulli numbers and Bernoulli polynomials

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \ \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

$\cdots$ up to $k = 10$ (Bernoulli expresses the right-hand side without factoring), he gives a general formula involving the numbers which are known today as Bernoulli numbers.

1. About Bernoulli numbers and Bernoulli polynomials

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- He claims that he did not take "a half of a quarter of an hour" to compute the sum of tenth powers of 1 to 1000, which he computed correctly as 91409924241424243424241924242500.
- Using modern notation, his formula is written as $\sum_{i=1}^{n} i^k = \sum_{j=0}^{k} \binom{k}{j} B_j \frac{n^{k+1-j}}{k+1-j}$, where $\binom{k}{j}$ is the binomial coefficient and B_j is the number determined by the recurrence formula

$$\sum_{j=0}^{k} \binom{k+1}{j} B_j = k+1, \ k = 0, 1, 2 \cdots . \tag{1}$$

It is this B_j that is subsequently called a Bernoulli number.

1. About Bernoulli numbers and Bernoulli polynomials

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- Japanese mathematician Seki Takakazu (1642-1708), published also posthumously, in 1712 (and thus 1 year before Bernoulli!), the formula for the sums of powers and the inductive definition of the Bernoulli numbers are given. His formula and definition are completely the same as Bernoulli's.

- Since this discovery, the Bernoulli numbers have appeared in many important results, including **the series expansions of trigonometric and hyperbolic trigonometric functions, the Euler-Maclaurin Summation Formula, the evaluation of the Riemann zeta function, and Fermat's Last Theorem**.

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- I mention here that a very extensive bibliography on Bernoulli numbers, compiled by Karl Dilcher, is available online: https://www.mathstat.dal.ca/ dilcher/

- A sequence of Bernoulli numbers B_0, B_1, B_2, … is given with the recurrence relation

$$(q+1)B_q = -\sum_{k=0}^{q-1} \binom{q+1}{k} B_k \tag{2}$$

where $B_0 = 1$.

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$$(q+1)B_q = -\sum_{k=0}^{q-1} \binom{q+1}{k} B_k \qquad (2)$$

where $B_0 = 1$.

- The first few Bernoulli numbers B_q are given as follows:

q	0	1	2	3	4	5	6	8	10	12	
B_q	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	$\frac{-1}{30}$	$\frac{5}{66}$	$\frac{-691}{2730}$	

- We have defined Bernoulli numbers by a recurrence formula. However, it is also common to define Bernoulli numbers using the generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$

Here $b_n = B_n$.

1. About Bernoulli numbers and Bernoulli polynomials

- The connection between Bernoulli polynomials and Bernoulli numbers is given with the relation

$$B_q(x) = \sum_{k=0}^{q} \binom{q}{k} B_k x^{q-k}. \tag{3}$$

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- The first few Bernoulli polynomials B_q are given as follows:

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \end{aligned}$$

- One of the most powerful applications of the Bernoulli numbers the evaluation of the Riemann zeta function.

Definition 1 (Riemann zeta function)

Let k be a real, $|k| \geq 1$. Then the Riemann zeta function over the real numbers, $\zeta(k)$, is defined as

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

- This function is important for many reasons, but we will highlight one result proven by Euler related to the prime numbers.

Theorem 2

For $k > 1$,

$$\zeta(k) = \prod_p \left(\frac{1}{1 - p^{-k}} \right)$$

over all primes p.

- The Bernoulli numbers help us to calculate the even values of this function.

Theorem 3

For any integer $k > 1$,

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}.$$

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- The Riemann zeta function is more famous as a complex function, with powers k in the complex plane.

1.1 Riemann Zeta Function and Bernoulli numbers

- In 1859, Bernhard Riemann (1826-1866)

hypothesized a result related to the complex Riemann zeta function, namely, that all of its nontrivial zeroes lie on the line $x = 1/2$.

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- Mathematicians and mathematical physicists have developed a whole branch of mathematics contingent on the fact that the hypothesis is true, so that anyone who manages to uncover the proof will immediately verify thousands of results.

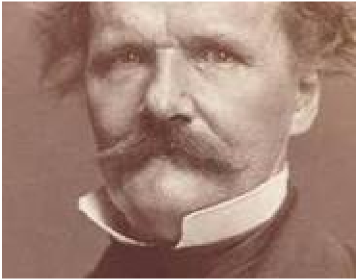
- Now, we look at an application of the Bernoulli numbers to one of the great solved problems of mathematics: the simply-stated Fermat's Last Theorem.

Theorem 4 (Fermat's last theorem)

The equation $x^n + y^n = z^n$ has no integer solutions x, y, z for positive integers $n > 2$.

1.2 Fermat's last theorem and Bernoulli numbers

- Ernst Kummer's result was the product of another mathematician's mistake. German mathematician Ernst Kummer (1810-1893)



had spent little time on Fermat's Last Theorem, which he considered a "curiosity of number theory rather than a major item," until March 1847, when the French mathematician Gabriel Lamé (1795-1870) published a "complete proof" of the theorem.

1.2 Fermat's last theorem and Bernoulli numbers

- Lamé's main contribution was noticing the sum $x^n + y^n$ could be decomposed into factors involving the n roots of unity:

$$(x+y)^n = (x+y)(x+\zeta y)(x+\zeta^2 y)\cdots(x+\zeta^{n-2}y)(x+\zeta^{n-1}y).$$

- This was a useful step; however, he incorrectly assumed that this factorization was unique in $\mathbb{Q}(\zeta_p)$. But Kummer himself had proven years prior that this was not the case. Kummer felt compelled to respond, and in the few weeks after Lamé's publication, he had a proof for a select group of integers n that would satisfy Fermat's Last Theorem. He called them the **"regular primes."**

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Odd prime p is a regular prime if the class number of $\mathbb{Q}(\zeta_p)$ is relatively prime to p.

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- Note that, by definition, a class number is the order of the ideal class group $\mathbb{Z}(\zeta_p)$. But more intuitively, the class number can be understood as "a scalar quantity describing how 'close' elements of a ring of integers are to having unique factorization".
- If the class number is 1, then the ring has unique factorization. For positive values greater than 1, the closer to 1 the class number is the 'closer' to having prime factorization.

- Kummer proved an equivalent definition, which almost by magic, involves the Bernoulli numbers:

Definition 6

A regular prime p is an integer such that it does not divide the numerator of $B_2, B_4, B_6, \cdots B_{p-3}$.

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- Kummer proved Fermat's Last Theorem for all regular primes.
- This, of course begs the question: how many regular primes are there?
- We know that the first irregular prime is 37, because

$$B_{32} = -\frac{7709321041217}{510} = 37 \times \frac{208360028141}{510} \qquad (4)$$

- Beyond that, we know that there are infinitely many irregular primes, but it is not known if there are infinitely many regular primes. Computational studies have shown that about %60 of primes are regular, and German mathematician Carl Ludwig Siegel (1896-1981) has conjectured that the exact proportion converges to $e^{-1/2}$. However, neither hypothesis has been confirmed.

- Regardless, Kummer's early work into Fermat's Last Theorem paved the way for mathematicians of the twentieth century to finish off the problem.

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- What is the formula for the following?

$$1^k + 2^k + 3^k + ... + n^k = ?$$

2. Bernoulli numbers/polynomials and consecutive power sums

- Now we need to introduce a family of numbers.
  In 1713, Jacob Bernoulli defined a sequence of Bernoulli numbers B_0, B_1, B_2, ... is given with the recurrence relation

$$(q+1)B_q = -\sum_{k=0}^{q-1} \binom{q+1}{k} B_k \qquad (5)$$

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- Now, by using the connection between Bernoulli polynomials and Bernoulli numbers we can give the following relation

$$1^k + 2^k + 3^k + ... + x^k = \frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1})$$

where $B_{k+1}(x+1)$ is $k+1$-st Bernoulli polynomial and B_{k+1} is $k+1$-st Bernoulli number.

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where $B_{k+1}(x+1)$ is $k+1$-st Bernoulli polynomial and B_{k+1} is $k+1$-st Bernoulli number.

- For example

$$1^6 + 2^6 + 3^6 + ... + x^6 = \frac{1}{7}(B_7(x+1) - B_7)$$

$$= \frac{1}{42}x(2x+1)(x+1)(3x^4 + 6x^3 - 3x + 1)$$

2. Bernoulli numbers/polynomials and consecutive power sums

- So, for example, we can calculate the following sum

$$1^6 + 2^6 + 3^6 + \ldots + 10^6 = \frac{1}{7}(B_7(11) - B_7) = 1978405$$

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3. Diophantine equations with power sums

- Now we consider the Diophantine equation

$$S_k(x) = y^n \tag{7}$$

where $n \geq 2,\ k, n, x, y \in \mathbb{Z}^+$ and

$$S_k(x) = 1^k + 2^k + \cdots + x^k. \tag{8}$$

- The first work on this equation was done in 1875. The classical question of Lucas was whether equation

$$1^2 + 2^2 + ... + x^2 = y^2 \tag{9}$$

has only the solutions $x = y = 1$ and $x = 24$, $y = 70$.

- The first work on this equation was done in 1875. The classical question of Lucas was whether equation

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has only the solutions $x = y = 1$ and $x = 24$, $y = 70$.

- In 1918, Watson proved that equation (9) has no solution other than $(x, y) = (1, 1)$ and $(24, 70)$.

- In 1956, Schäffer gave important results on the equation

$$1^k + 2^k + \cdots + x^k = y^n.$$

So, this equation is called "Schäffer's equation".

Lemma 1 (Schäffer, 1956)

If $k = 1$, then $S_1(x) = \frac{x(x+1)}{2}$. While, if $k \neq 1$, we can write

$$S_k(x) = \begin{cases} \dfrac{x^2(x+1)^2 R_k(x)}{C_k}, & \text{if } k > 1 \text{ odd}, \\ \\ \dfrac{x(x+1)(2x+1)R_k(x)}{C_k}, & \text{if } k \geq 2 \text{ even}. \end{cases} \tag{10}$$

($C_k > 0$, $C_k \in \mathbb{Z}$ and $R_k(x)$ is a polynomial with integer coefficient)

He proved the following:

Theorem 2 (Schäffer, 1956)

For fixed $k \geq 1$ and $n \geq 2$, the eq. (7) has at most finitely many solutions in positive integers x and y, unless

$$(k, n) \in \{(1, 2), (3, 2), (3, 4), (5, 2)\}, \tag{11}$$

where, in each case, there are infinitely many such solutions.

- Schäffer proved that the eq. (7) has finitely many solutions in the each following cases.

$k \in \{1, 3, 5\}$	$n = 4$
$k = 3$	$n = 8$
$k \in \{4, 6, 8, 9, 10\}$	$n = 2$
$k \leq 11$	$n \in \{3, 5\}$
$k \leq 11,\ k \neq 10$	$n \in \{29, 41, 53, 113, 173, 281, 509, 641\}$

- Schäffer proved that the eq. (7) has finitely many solutions in the each following cases.

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$k \in \{4, 6, 8, 9, 10\}$	$n = 2$
$k \leq 11$	$n \in \{3, 5\}$
$k \leq 11,\ k \neq 10$	$n \in \{29, 41, 53, 113, 173, 281, 509, 641\}$

- In the terminology of his work,
$(x, y) = (1, 1)$ was called as "trivial solution", $(x, y) = (24, 70)$ was called as "non-trivial solution" with $(k, n) = (2, 2)$.

Schäffer gave the following conjecture:

Conjecture 1 (Schäffer,1956)

Let $k \geq 1$ be fixed and let $n \geq 2$ be positive integers with (k, n) not in the above list. Then the eq. (7) has only non-trivial solution $(k, n, x, y) = (2, 2, 24, 70)$ tir.

Schäffer's proof used an ineffective method due to Thue and Siegel so his result is also ineffective. This means that the proof does not provide any algorithm to find all solutions.
Applying Baker's method, Győry, Tijdeman and Voorhoeve proved a more general and effective result in which the exponent n is also unknown.

Theorem 3 (Győry, Tijdeman and Voorhoeve, 1980)

Let $k \geq 2$ and r be fixed integers with $k \notin \{3, 5\}$ if $r = 0$, and let s be a square-free odd integer. Then the equation

$$s(1^k + 2^k + ... + x^k) + r = y^n$$

in positive integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions and all these can be effectively determined.

Of particular importance is the special case when $s = 1$ and $r = 0$.

Corollary 4 (Győry, Tijdeman and Voorhoeve, 1980)

For given $k \geq 2$ with $k \notin \{3, 5\}$, equation (7) has only finitely many solutions in integers $x, y \geq 1$, $n \geq 2$, and all these can be effectively determined.

The following striking result is due to Voorhoeve, Győry and Tijdeman:

Theorem 5 (Voorhoeve, Győry and Tijdeman, 1979)

Let $R(x)$ be a fixed polynomial with integer coefficients and let $k \geq 2$ be a fixed integer such that $k \notin \{3, 5\}$. Then the equation

$$1^k + 2^k + ... + x^k + R(x) = by^n$$

in integers $x, y \geq 2$, $n \geq 2$ has only finitely many solutions, and an effective upper bound can be given for n.

Later, various generalizations and analogues of Győry, Tijdeman and Voorhoeve have been established by several authors (Brindza, Pintér, Dilcher, Urbanowicz, Kano $\cdots$). For a survey of these results we refer to the paper of Győry and Pintér and the references given there.

- In the last 50 years, the various generalizations of Schäffer's equation were considered, but the none of them couldn't do any progress on the conjecture. The first progress was recorded in 2003 with the following result:

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Theorem 6 (Jacobson, Pintér and Walsh, 2003)

For $n = 2$ and even values of k with $k \leq 58$, eq. (7) has only the trivial solution except in the case $k = 2$, when there is the anomalous solution $(x, y) = (24, 70)$.

- Finding all solutions of the eq. (7) is a hard problem because n is not fixed. Next year, the following nice result was given by Bennett, Győry and Pintér:

- Finding all solutions of the eq. (7) is a hard problem because n is not fixed. Next year, the following nice result was given by Bennett, Györy and Pintér:

Theorem 7 (Bennett, Győry and Pintér, 2004)

For $1 \leq k \leq 11$ *and* (k, n) *not in the set* (11), *equation* (7) *has only the trivial solution, unless* $k = 2$, *in which case there is the additional solution* $(n, x, y) = (2, 24, 70)$.

4 years later, using several currently available techniques, including Baker's method, Frey curves and modular forms, Pintér gave the following result about Schäffer's conjecture:

Theorem 8 (Pintér, 2007)

For odd values of k*, with* $1 \leq < k < 170$*, the equation*

$$S_k(x) = y^{2n}, \text{ in positive integers } x, y, n \text{ with } n > 2$$

possesses only the trivial solution $(x, y) = (1, 1)$.

- In 2015, a new progress on Schäffer's conjecture was recorded by Hajdu. We first recall that $v_p(N)$ stands for the exponent of the prime p in the prime factorization of the positive integer N.

- In 2015, a new progress on Schäffer's conjecture was recorded by Hajdu. We first recall that $v_p(N)$ stands for the exponent of the prime p in the prime factorization of the positive integer N.
- Hajdu gave the following lemmas for his main theorem:

Lemma 9 (Hajdu, 2015)

Let x be a positive integer. Then we have

$$v_2(S_k(x)) = \begin{cases} \upsilon_2(x(x+1)) - 1, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2\upsilon_2(x(x+1)) - 2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Lemma 10 (Hajdu, 2015)

Let x be a positive integer. Then we have

$$v_3(S_k(x)) = \begin{cases} \upsilon_3(x(x+1)), & \textit{if} \quad k=1, \\ \upsilon_3(x(x+1)(2x+1))-1, & \textit{if} \quad k \textit{ is even}, \\ 0, & \textit{if} \quad x \equiv 1 \pmod 3 \quad \textit{and} \quad k \geq 3 \textit{ is odd} \\ \upsilon_3(kx^2(x+1)^2)-1, & \textit{if } x \equiv 0,2 \pmod 3 \quad \textit{and} \\ k \geq 3 \textit{ is odd.} & \end{cases}$$

Finally, by using the aboved Lemmas, Hajdu gave the following result about Schäffer's conjecture:

Theorem 11 (Hajdu, 2015)

Suppose that $x \equiv 0, 3 \pmod 4$ *and* $x < 25$. *Then the eq.* (7) *has only known solutions.*

The next year, these results were extended by Bérczes, Hajdu, Miyazaki and Pink.

Theorem 12 (Bérczes, Hajdu, Miyazaki and Pink, 2016)

All solutions of equation (7) in positive integers k, n, x, y with $x < 25$ and $n \geq 3$ are given by

$$(k, n, x, y) = (k, n, 1, 1), (3, 4, 8, 6).$$

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Theorem 12 (Bérczes, Hajdu, Miyazaki and Pink, 2016)

All solutions of equation (7) in positive integers k, n, x, y with $x < 25$ and $n \geq 3$ are given by

$$(k, n, x, y) = (k, n, 1, 1), (3, 4, 8, 6).$$

As a simple consequence they obtain the following immediate:

Corollary 13 (Bérczes, Hajdu, Miyazaki and Pink, 2016)

For $x < 25$ and $n \geq 3$, Schaffer's conjecture is true.

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- Now we consider the eq.

$$T_{k,\ell}(x) = y^n \tag{12}$$

where

$$T_{k,\ell}(x) = (x+1)^k + (x+2)^k + \ldots + (\ell x)^k, \;\; k, \ell \in \mathbb{Z}^+ \tag{13}$$

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- In 2013, Zhang and Bai worked the eq. (12) for the case $k = \ell = 2$ and they gave the following:

Theorem 14 (Bai and Zhang, 2013)

For $n > 1$ all solutions of the eq. (12) are $(x, y) = (0, 0), (x, y, n) = (1, \mp 2, 2), (2, \mp 5, 2), (24, \mp 182, 2)$ or for $2 \nmid n$ the only solution is $(x, y) = (-1, -1)$.

3.1 Old results on a Diophantine equation with power sums

Theorem 15 (Soydan, 2017)

Let $k, \ell \geq 2$ *fixed integers. Then all solutions of the equation* $(x+1)^k + (x+2)^k + ... + (\ell x)^k = y^n$ *in integers* $x, y \geq 1$ *and* $n \geq 2$ *satisfy* $n < C_1$ *where* C_1 *is an effectively computable constant depending only on* ℓ *and* k.

Theorem 15 (Soydan, 2017)

Let $k, \ell \geq 2$ fixed integers. Then all solutions of the equation $(x+1)^k + (x+2)^k + ... + (\ell x)^k = y^n$ in integers $x, y \geq 1$ and $n \geq 2$ satisfy $n < C_1$ where C_1 is an effectively computable constant depending only on ℓ and k.

Theorem 16 (Soydan, 2017)

Let $k, \ell \geq 2$ fixed integers such that $k \neq 3$. Then all solutions of the equation $(x+1)^k + (x+2)^k + ... + (\ell x)^k = y^n$ in integers x, y, n with $x, y \geq 1$, $n \geq 2$, and $\ell \equiv 0 \pmod 2$ satisfy $\max\{x, y, n\} < C_2$ where C_2 is an effectively computable constant depending only on ℓ and k.

Theorem 15 (Soydan, 2017)

Let $k, \ell \geq 2$ fixed integers. Then all solutions of the equation $(x+1)^k + (x+2)^k + \ldots + (\ell x)^k = y^n$ in integers $x, y \geq 1$ and $n \geq 2$ satisfy $n < C_1$ where C_1 is an effectively computable constant depending only on ℓ and k.

Theorem 16 (Soydan, 2017)

Let $k, \ell \geq 2$ fixed integers such that $k \neq 3$. Then all solutions of the equation $(x+1)^k + (x+2)^k + \ldots + (\ell x)^k = y^n$ in integers x, y, n with $x, y \geq 1$, $n \geq 2$, and $\ell \equiv 0 \pmod 2$ satisfy $\max\{x, y, n\} < C_2$ where C_2 is an effectively computable constant depending only on ℓ and k.

- By Theorem 16, it was proved that this equation has finitely many solutions where $k \neq 1, 3$, ℓ is even, $n \geq 2$ and $x, y, k, n \in \mathbb{Z}^+$ and it has infinitely many solutions where $n = 2$ and $k = 1, 3$.

- We need some lemmas for proving Theorems 15 - 16

Lemma 7

$(x+1)^k + (x+2)^k + \dots + (\ell x)^k = \dfrac{B_{k+1}(\ell x + 1) - B_{k+1}(x+1)}{k+1}$ *where*

$$B_q(x) = x^q - \frac{1}{2} q x^{q-1} + \frac{1}{6}\binom{q}{2} x^{q-2} + \dots = \sum_{i=0}^{q} \binom{q}{i} B_i x^{q-i}$$

is the q-th Bernoulli polynomial.

Lemma 8 (Brindza, 1984)

Let $H(x) \in \mathbb{Q}[x]$,

$$H(x) = a_0 x^N + \dots + a_N = a_0 \prod_{i=1}^{n} (x - \alpha_i)^{r_i},$$

with $a_0 \neq 0$ *and* $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Let* $b \neq 0 \in \mathbb{Z}$, $2 \leq m \in \mathbb{Z}$ *and define* $t_i = \frac{m}{(m, r_i)}$. *Suppose that* $\{t_1, \dots t_n\}$ *is not a permutation of the n-tuples*
(*a*) $\{t, 1, \dots, 1\}$, $t \geq 1$;
(*b*) $\{2, 2, 1, \dots, 1\}$ *Then all solutions* $(x, y) \in \mathbb{Z}^2$ *of the equation*

$$H(x) = by^m$$

satisfy $\max\{|x|, |y|\} < C$, *where* C *is effectively computable constant depending only on* H, b *and* m.

Lemma 9 (Schinzel & Tijdeman, 1976)

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least 2 distinct roots. Then there exists an effective constant $N(f)$ such that any solution of the equation $f(x) = y^n$ in $x, n \in \mathbb{Z}$, $y \in \mathbb{Q}$ satisfies $n \leq N(f)$.

Lemma 9 (Schinzel & Tijdeman, 1976)

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least 2 distinct roots. Then there exists an effective constant $N(f)$ such that any solution of the equation $f(x) = y^n$ in $x, n \in \mathbb{Z}$, $y \in \mathbb{Q}$ satisfies $n \leq N(f)$.

Lemma 10 (Schinzel-Tijdeman, 1976)

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial having at least 3 simple roots. Then the equation $f(x) = y^n$ has at most finitely many solutions in $x, n \in \mathbb{Z}$, $y \in \mathbb{Q}$ satisfying $n > 1$. If $f(x)$ has 2 simple roots then the equation $f(x) = y^n$ has only finitely many solutions with $n > 2$. In both cases the solutions can be explicitly determined.

- Now we need two key lemmas:

Lemma 11 (Soydan, 2017)

For $k \in \mathbb{Z}^+$ let $B_k(x)$ be the k-th Bernoulli polynomial. Then the polynomial

$$G(x) = \frac{B_{k+1}(\ell x + 1) - B_{k+1}(x+1)}{k+1}$$

has at least two distinct zeros where $G(x) = y^n$.

Lemma 12 (Soydan, 2017)

For $q \geq 2$ let $B_q(x)$ be the q-th Bernoulli polynomial. Let

$$P(x) = B_q(\ell x + 1) - B_q(x + 1) \tag{14}$$

where ℓ is even. Then
(i) $P(x)$ has at least three zeros of odd multiplicity unless $q \in \{2, 4\}$.
(ii) For any odd prime p, at least two zeros of $P(x)$ have multiplicities relatively prime to p.

The idea for the proof of Theorem 15:

Let $x, y \geq 1$ and $n \geq 2$ be an arbitrary solution of the equation

$$T_{k,\ell}(x) = y^n \tag{15}$$

where

$$T_{k,\ell}(x) = (x+1)^k + (x+2)^k + \ldots + (\ell x)^k,\ k, \ell \in \mathbb{Z}^+.$$

in integers. We know from Lemma 11 that $T_{k,\ell}(x)$ has at least two distinct zeros. Hence it follows from the equation (15) by applying Lemma 9 (Schinzel & Tijdeman, 1976) that we get an effective bound for n.

The idea for the proof of Theorem 16:

We know from Theorem 15 that n is bounded, i.e. $n < C_1$ with an effectively computable C_1. So we may assume that n is fixed. Using Lemma 8 (Brindza, 1984) and Lemma 12 (Soydan, 2017), we can prove the rest of the part of the theorem.

3.2 New results on a Diophantine equation with power sums

- Consider $T_{k,\ell}(x) = (x+1)^k + (x+2)^k + \cdots + (\ell x)^k$. We have $T_{k,\ell}(x) = B_{k+1}(\ell x + 1) - B_{k+1}(x+1)$, where

$$B_q(x) = x^q - \frac{1}{2} q x^{q-1} + \frac{1}{6}\binom{q}{2} x^{q-2} + \cdots = \sum_{i=0}^{q} \binom{q}{i} x^{q-i} B_i$$

is the q-th Bernoulli polynomial with $q = k+1$. Therefore,

- Consider $T_{k,\ell}(x) = (x+1)^k + (x+2)^k + \cdots + (\ell x)^k$. We have $T_{k,\ell}(x) = B_{k+1}(\ell x + 1) - B_{k+1}(x+1)$, where

$$B_q(x) = x^q - \frac{1}{2}qx^{q-1} + \frac{1}{6}\binom{q}{2}x^{q-2} + \cdots = \sum_{i=0}^{q}\binom{q}{i}x^{q-i}B_i$$

is the q-th Bernoulli polynomial with $q = k+1$. Therefore,

- $$T_{k,\ell}(x) = \sum_{i=0}^{k+1}\binom{k+1}{i}(\ell x+1)^{k+1-i}B_i - \sum_{i=0}^{k+1}\binom{k+1}{i}(x+1)^{k+1-i}B_i =$$

$$(\ell^{k+1}-1)x^{k+1} + \frac{(k+1)}{2}(\ell^k-1)x^k + \frac{(k+1)k}{12}(\ell^{k-1}-1)x^{k-1} + \cdots .$$

Note that $T_{k,\ell}(0) = 0$ and the multiplicity of 0 as root of $T_{k,\ell}(x)$ is 1 if $k+1$ is odd and 2 if $k+1$ is even.

Proposition 17 (Bartoli and Soydan, 2020)

The polynomial $T_{k,\ell}(x)$ has at least three distinct roots.

3.2 New results on a Diophantine equation with power sums

Proposition 17 (Bartoli and Soydan, 2020)

The polynomial $T_{k,\ell}(x)$ has at least three distinct roots.

Proof.

Let 0 be a root of multiplicity $r = 1, 2$ of $T_{k,\ell}(x)$ and suppose that $T_{k,\ell}(x)$ has only two distinct roots. Then

$$\frac{T_{k,\ell}(x)}{\ell^{k+1} - 1} = x^r (x + \alpha)^{k+1-r}$$

for some α. This means that

$$\alpha(k+1-r) = \frac{(k+1)(\ell^k - 1)}{2(\ell^{k+1} - 1)}, \qquad \alpha^2 \binom{k+1-r}{2} = \frac{(k+1)k(\ell^{k-1} - 1)}{12(\ell^{k+1} - 1)}.$$

From here, using some inequalities, we get a contradiction. So the proof is completed. □

Theorem 18 (Bartoli and Soydan, 2020)

Let k, ℓ be fixed integers such that $k \geq 2$, $k \neq 3$, $\ell \geq 2$. Then all solutions of equation

$$(x+1)^k + (x+2)^k + ... + (\ell x)^k = y^n \tag{16}$$

in integers x, y, n with $x, y \geq 1$, $n \geq 2$ satisfy $\max\{x, y, n\} < C$ where C is an effectively computable constant depending only on ℓ and k.

3.2.1 Sketch for the Proof of Theorem 18

We distinguish the cases $k+1$ *odd* and $k+1$ *even*.

Case 1: We suppose that $k+1$ is odd and then the multiplicity of the root 0 is $r=1$. Then $t_0 = \frac{n}{(n,1)} = n$.

Also, using that $\sum_{i=0}^{k-1} \binom{k}{i} B_i = 0$, the term of degree of 1 of $T_{k,\ell}(x)$ is

$$(\ell-1)\sum_{i=0}^{k}\binom{k+1}{i}(k+1-i)B_i$$

$$=(k+1)(\ell-1)\sum_{i=0}^{k}\binom{k}{i}B_i$$

$$=(k+1)(\ell-1)B_k \neq 0,$$

where B_i is i-th Bernoulli number.

3.2.1 Sketch for the Proof of Theorem 18

(i) Suppose $n \nmid k$.
Since k is even and $n \nmid k$, the case $n = 2$ is impossible. Therefore $n > 2$ since k is even and then there exists at least one root distinct from 0 such that $n \nmid r_i$, where r_i is its multiplicity. This yields $t_i = \frac{n}{(n, r_i)} \neq 1$ and therefore the bad patterns in Lemma 8 [Brindza, 1984] are avoided.
(ii) Suppose $n \mid k$.
If all the roots of the polynomial $T_{k,\ell}(x)$ have multiplicity r_i divisible by n, then $T_{k,\ell}(x)/x = (\ell^{k+1} - 1)f(x)^n$, where $f(x) = x^s + \sum_{i=0}^{s-1} \alpha_i x^i$, with $k = ns$. Since all coefficients of $T_{k,\ell}(x)/(x(\ell^{k+1} - 1))$ are rational, $f(x)$ also must have rational coefficients. So the term α_0 is rational and $\alpha_0^n = (k+1)(\ell - 1)B_k/(\ell^{k+1} - 1)$.
According to **the von Staudt-Clausen theorem**, if $B_k \neq 0$ then 2 divides the denominator but 4 does not divide. In this case, if 2^a is the highest power that divides $\ell - 1$, then 2^a is the highest power which also divides $\ell^{k+1} - 1$. Therefore 2 divides and 4 does not divide the denominator of α_0^n which is a contradiction.

If there exists at least one root having multiplicity r_i not divisible by n, then the pattern does not correspond to $(n, 1, 1, 1, 1\ldots)$. So this case is completed.

Case 2: Now suppose that $k+1$ is even and then the multiplicity of the root 0 is $r = 2$. Then $t_0 = \frac{n}{(n,2)} \in \{n/2, n\}$. Also, $B_{k-1} \neq 0$ and the term of degree 2 in $T_{k,\ell}(x)$ is given by

$$(\ell^2 - 1)\sum_{i=0}^{k-1}\binom{k+1}{i}\binom{k+1-i}{2}B_i = \binom{k+1}{2}(\ell^2-1)\sum_{i=0}^{k-1}\binom{k-1}{i}B_i$$
$$= \binom{k+1}{2}(\ell^2-1)B_{k-1} \neq 0.$$

3.2.1 Sketch for the Proof of Theorem 18

(i) Suppose $n \mid (k-1)$.
If there exists at least one root having multiplicity r_i not divisible by n, then the pattern does not correspond to $(n, 1, 1, 1, 1, \ldots)$.
If all the roots of the polynomial $T_{k,\ell}(x)$ have multiplicity r_i divisible by n, then $T_{k,\ell}(x)/(x^2(\ell^{k+1}-1))$ must be a monic polynomial which is also an n-power, then $T_{k,\ell}(x)/x^2 = (\ell^{k+1}-1)f(x)^n$, where $f \in \mathbb{Q}[x]$.
By **the von Staudt-Clausen theorem** again, a prime p divides the denominator of B_{k-1} if and only if $(p-1) \mid (k-1)$ and the denominator is square-free.
Suppose that $2^e \parallel (k+1)/2$, that is $2^e \mid (k+1)/2$ and $2^{e+1} \nmid (k+1)/2$.
Then

$$\frac{k+1}{2} \equiv 2^e \pmod{2^{e+1}}.$$

Now assume that ℓ is odd.

Then

$$\ell^2 = 1 + 8t \pmod{2^{e+1}},$$

therefore

$$\frac{\ell^2 - 1}{\ell^{k+1} - 1} = \frac{1}{\ell^{k-1} + \ell^{k-3} + \ell^{k-5} + \cdots + \ell^2 + 1} = \frac{1}{z},$$

where

$$\begin{aligned} z &\equiv 1 + (1+8t) + (1+8t)^2 + (1+8t)^3 + \cdots + (1+8t)^{(k-1)/2} \\ &\equiv \frac{(1+8t)^{(k+1)/2} - 1}{8t} \pmod{2^{e+1}} \\ &\equiv \frac{\frac{k+1}{2}8t + \frac{k^2-1}{8}(8t)^2 + \cdots}{8t} \pmod{2^{e+1}} \\ &\equiv \cdots \equiv \frac{k+1}{2} \pmod{2^{e+1}} \equiv 2^e \pmod{2^{e+1}}. \end{aligned}$$

Thus $2^e \mid\mid z$ and then 2 is the highest power of 2 dividing the denominator of

$$\binom{k+1}{2}\frac{\ell^2-1}{\ell^{k+1}-1}B_{k-1}.$$

This is not possible since

$$\alpha_0^n=\binom{k+1}{2}\frac{\ell^2-1}{\ell^{k+1}-1}B_{k-1}.$$

Since the case when ℓ is even for the equation (15) has already been considered in [Soydan, 2017], the proof of case (i) is completed.

(ii) Suppose $n \nmid (k-1)$. Then n must be at least 3, since $k-1$ is even.

If $n=3$, then $t_0 = \frac{n}{(n,2)} = 3$ and there exists at least one root distinct from 0 such that $n \nmid r_i$, where r_i is its multiplicity. This yields $t_i = \frac{n}{(n,r_i)} \neq 1$ and therefore the bad patterns are avoided.

If $n=4$, then $t_0 = \frac{n}{(n,2)} = 2$. Since $n \nmid k-1$, it can be still possible that there exists a unique root of multiplicity r_i, not divisible by 4, but divisible by 2, and all the other multiplicities are divisible by 4. So we can write $T_{k,\ell}(x)/x^2 = (\ell^{k+1}-1)f(x)^2$ where $f \in \mathbb{Q}[x]$, since $T_{k,\ell}(x)$ has, apart from 0, one root of multiplicity 2, and all the other multiplicities are divisible by 4.

Here we distinguish two cases. First we suppose that ℓ is odd. Then, following the steps in Case 2 (i), we get that 2 is the highest power of 2 dividing the denominator of

$$\binom{k+1}{2}\frac{\ell^2-1}{\ell^{k+1}-1}B_{k-1},$$

which contradicts with

$$\alpha_0^2=\binom{k+1}{2}\frac{\ell^2-1}{\ell^{k+1}-1}B_{k-1}.$$

The case when ℓ is even has been considered in [Soydan, 2017]. So the proof of the case (ii) with $n=4$ is completed.

If $n > 4$, then $t_0 = \frac{n}{(n,2)} > 2$, and there exists at least one root distinct from 0 such that $n \nmid r_i$, where r_i is its multiplicity. This yields $t_i = \frac{n}{(n,r_i)} \neq 1$ and therefore the bad patterns are avoided. This finishes the proof of the theorem.

- Now, we are interested in the integer solutions of the eq.

$$T_k(x) = y^n \tag{17}$$

where

$$T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k. \tag{18}$$

- Now, we are interested in the integer solutions of the eq.

$$T_k(x) = y^n \tag{17}$$

where

$$T_k(x) = (x+1)^k + (x+2)^k + ... + (2x)^k. \tag{18}$$

- By Theorem 16, this eq. has finitely many solutions.
Here we first provide upper bounds for the exponent n in equation (17) in terms of 2 and 3-valuations v_2 and v_3 of some functions of x and x, k.

Theorem 19 (Bérczes, Pink, Savaș and Soydan, 2018)

(*i*) *Assume first that* $x \equiv 0 \pmod 4$. *Then for any solution* (k, n, x, y) *of equation* (17), *we get*

$$n \leq \begin{cases} v_2(x) - 1, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2v_2(x) - 2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

(ii)*Assume that* $x \equiv 1 \pmod 4$ and $k = 1$, then for any solution (k, n, x, y) of equation (17), we get $n \le v_2(3x+1) - 1$.
Suppose next that $x \equiv 1, 5 \pmod 8$ and $x \not\equiv 1 \pmod{32}$ with $k \neq 1$.
Then for any solution (k, n, x, y) of equation (17), we get

$$n \le \begin{cases} v_2(7x+1)-1, & \text{if } x \equiv 1 \pmod 8 \text{ and k=2,} \\ v_2((5x+3)(3x+1))-2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k = 3, \\ v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \ge 3 \text{ is odd,} \\ 1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \ge 2 \text{ is even,} \\ 2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \ge 4 \text{ is even,} \\ 3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \ge 5 \text{ is odd} \\ & \text{or} \\ & \text{if } x \equiv 17 \pmod{32} \text{ and } k \ge 4 \text{ is even,} \\ 4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \ge 5 \text{ is odd.} \end{cases}$$

(iii)*Suppose now that* x $\equiv 0 \pmod 3$ and k is odd or $x \equiv 0, 4 \pmod 9$ and $k \geq 2$ is even. Then for any solution (k, n, x, y) of equation (17),

$$n \leq \begin{cases} v_3(x), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 1, \\ v_3(x) - 1, & \text{if } x \equiv 0 \pmod 9 \text{ and } k \geq 2 \text{ is even}, \\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd}, \\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 3, \\ v_3(2x+1) - 1, & \text{if } x \equiv 4 \pmod 9 \text{ and } k \geq 2 \text{ is even}. \end{cases}$$

Combining effective upper bounds are concerning n and Baker's theory (using M. Laurent's results) we have following results:

Theorem 20 (Bérczes, Pink, Savaș and Soydan, 2018)

Assume that $x \equiv 1, 4 \pmod 8$ *or* $x \equiv 4, 5 \pmod 8$*. Then Eq.* (17) *has no solution with* $k = 1$ *or* $k \geq 2$ *is even, respectively.*

Theorem 21 (Bérczes, Pink, Savaș and Soydan, 2018)

Consider equation (17) *in positive integer unknowns* (x, k, y, n) *with* $2 \leq x \leq 13, k \geq 1, y \geq 2$ *and* $n \geq 3$*. Then equation* (17) *has no solutions.*

5.2. Properties of polynomial $T_k(x)$

Lemma 22 (Bérczes, Pink, Savaș and Soydan, 2018)

$$T_k(x) = \frac{1}{k+1}\left(B_{k+1}(2x+1) - B_{k+1}(x+1)\right) \tag{19}$$

Now we give a usefull Lemma about the polynomial $T_k(x)$.

Lemma 23 (Bérczes, Pink, Savaş and Soydan, 2018)

If $k=1$, then $T_1(x)=\frac{x(3x+1)}{2}$, while for $k>1$ we can write
(i) $T_k(x)=\frac{1}{D_k}x(2x+1)M_k,\quad$ *if $k\geq 2$ is even,*
(ii) $T_k(x)=\frac{1}{D_k}x^2(3x+1)M_k,\quad$ *if $k>1$ is odd*
where D_k is a positive integer and $M_k(x)$ is a polynomial with integer coefficients.

Proof.

It is used the fact that Bernoulli polynomials are Appell polynomials. □

Lemma 24 (Sondow-Tsukerman, 2014)

If p is a prime, $d,\ q \in \mathbb{N}$, $k \in \mathbb{Z}^+$, $m_1 \in p^d\mathbb{N} \cup \{0\}$ and $m_2 \in p^d\mathbb{N} \cup \{0\}$, then

$$S_k(qm_1 + m_2) \equiv qS_k(m_1) + S_k(m_2) \pmod{p^d}. \tag{20}$$

Lemma 25 (Sondow-Tsukerman, 2014)

Let p be an odd prime and let m and k be positive integers.
(i) *For some integer $d \geq 1$, we can write*

$$m = qp^d + r\frac{p^d - 1}{p - 1} = qp^d + rp^{d-1} + rp^{d-2} + \cdots + rp^0,$$

where $r \in \{0, 1, ..., p-1\}$ and $0 \leq q \not\equiv r \equiv m \pmod{p}$.
(ii) *In the case of $m \equiv 0 \pmod{p}$, we have*

$$S_k(m) \equiv \begin{cases} -p^{d-1} \pmod{p^d}, & \text{if } p-1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p-1 \nmid k. \end{cases}$$

5.2. Congruence properties of $S_k(x)$

(iii)*In the case of* m $\equiv -1 \pmod{p}$, we have

$$S_k(m) \equiv \begin{cases} -p^{d-1}(q+1) \pmod{p^d}, & \text{if } p-1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p-1 \nmid k. \end{cases}$$

(iv) In the case of $m \equiv \frac{p-1}{2} \pmod{p}$, we have

$$S_k(m) \equiv \begin{cases} -p^{d-1}(q+\frac{1}{2}) \pmod{p^d}, & \text{if } p-1 \mid k, \\ 0 \pmod{p^d}, & \text{if } p-1 \nmid k. \end{cases}$$

5.4. Linear forms in logarithms

- For an algebraic number α of degree d over $\mathbb{Q}$, we define the *absolute logarithmic height* of α by the following formula:

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d}\log\max\{1, |\alpha^{(i)}|\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

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where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

- Let α_1 and α_2 be multiplicatively independent algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. Consider the linear form in two logarithms:

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where $\log \alpha_1, \log \alpha_2$ are any determinations of the logarithms of α_1, α_2 respectively, and b_1, b_2 are positive integers.
We shall use the following result due to Laurent:

Lemma 26 (Laurent, 2008)

Let ρ *and* μ *be real numbers with* $\rho > 1$ *and* $1/3 \leq \mu \leq 1$. *Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda = \sigma \log \rho.$$

Let a_1, a_2 *be real numbers such that*

$$a_i \geq \max \{1, \rho |\log \alpha_i| - \log |\alpha_i| + 2D\mathrm{h}(\alpha_i)\} \qquad (i = 1, 2),$$

where

$$D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}] \,.$$

Let h *be a real number such that*

$$h \geq \max \left\{ D \left(\log \left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) + \log \lambda + 1.75 \right) + 0.06, \ \lambda, \ \frac{D \log 2}{2} \right\}.$$

We assume that

$$a_1 a_2 \geq \lambda^2.$$

Put

$$H = \frac{h}{\lambda} + \frac{1}{\sigma}, \quad \omega = 2 + 2\sqrt{1 + \frac{1}{4H^2}}, \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Then we have

$$\log |\Lambda| \geq -Ch'^2 a_1 a_2 - \sqrt{\omega\theta}\, h' - \log\left(C' h'^2 a_1 a_2\right)$$

with

$$h' = h + \frac{\lambda}{\sigma}, \quad C = C_0 \frac{\mu}{\lambda^3 \sigma}, \quad C' = \sqrt{\frac{C\sigma\omega\theta}{\lambda^3 \mu}},$$

where

$$C_0 = \left(\frac{\omega}{6} + \frac{1}{2}\sqrt{\frac{\omega^2}{9} + \frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{a_1 a_2}\,H^{1/2}} + \frac{4}{3}\left(\frac{1}{a_1} + \frac{1}{a_2}\right)\frac{\lambda\omega}{H}} \right)^2.$$

- Let $A = \{2, 3, 6, 7, 10, 11\}$ and consider equation (17) with $x \in A$. The following lemma provides sharp upper bounds for the solutions n, k of the equation (17) and will be used in the proof of main theorem.

Lemma 27 (Berczes, Pink, Savaș and Soydan, 2018)

Let $A = \{2, 3, 6, 7, 10, 11\}$ *and consider equation* (17) *with* $x \in A$ *in integer unknowns* (k, y, n) *with* $k \geq 83, y \geq 2$ *and* $n \geq 3$ *a prime. Then for* $y > 4x^2$ *we have* $n \leq n_0$*, for* $y > 10^6$ *even* $n \leq n_1$ *holds, and for* $y \leq 4x^2$ *we have* $k \leq k_1$*, where* $n_0 = n_0(x), n_1 = n_1(x)$ *and* $k_1 = k_1(x)$ *are given in the following table.*

x	n_0 $(y > 4x^2)$	n_1 $(y > 10^6)$	k_1 $(y \leq 4x^2)$
2	7,500	3,200	45,000
3	21,000	10,000	120,000
6	94,000	53,000	540,000
7	128,000	74,200	740,000
10	253,000	157,000	1,450,000
11	301,000	190,000	1,750,000

Table: *Bounding n and k under the indicated conditions*

Proof.

We distinguish three cases: $y > 4x^2$, $y > 10^6$, $y \leq 4x^2$. The main tool on the proof is Lemma 26 (Laurent-2008) and all computations are supported by MAGMA. □

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Lemma 28 (Bérczes, Pink, Savaș and Soydan, 2018)

For $q, k, t \geq 1$ and $q \equiv 1 \pmod 2$, we have

$$v_2(T_k(2^t q)) = \begin{cases} t-1, & \text{if } k = 1 \text{ or } k \text{ is even,} \\ 2t-2, & \text{if } k \geq 3 \text{ is odd.} \end{cases}$$

Proof.

On the proof, the method in Macmillian-Sondow 2012 (Lemma-1) is used. □

5.5. Formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$

Lemma 29 (Bérczes, Pink, Savaș and Soydan, 2018)

(i) Let x be a positive even integer. Then we have,

$$v_2(T_k(x)) = \begin{cases} v_2(x) - 1, & \text{if } k = 1 \text{ or } k \text{ is even}, \\ 2v_2(x) - 2, & \text{if } k \geq 3 \text{ is odd}. \end{cases}$$

(ii) Let x be a positive odd integer. If x is odd and $k = 1$, then for any solution (k, n, x, y) of (17) we get $v_2(T_k(x)) = v_2(3x + 1) - 1$.

5.5. Formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$

If $x \equiv 1, 5 \pmod 8$ and $x \not\equiv 1 \pmod{32}$ with $k \neq 1$, then we have $v_2(T_k(x))$

$$
= \begin{cases}
v_2(7x+1)-1, & \text{if } x \equiv 1 \pmod 8 \text{ and k=2},\\
v_2((5x+3)(3x+1))-2, & \text{if } x \equiv 1 \pmod 8 \text{ and } k = 3,\\
v_2(3x+1), & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 3 \text{ is odd},\\
1, & \text{if } x \equiv 5 \pmod 8 \text{ and } k \geq 2 \text{ is even},\\
2, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 4 \text{ is even},\\
3, & \text{if } x \equiv 9 \pmod{16} \text{ and } k \geq 5 \text{ is odd}\\
 & \text{or}\\
 & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 4 \text{ is even},\\
4, & \text{if } x \equiv 17 \pmod{32} \text{ and } k \geq 5 \text{ is odd}.
\end{cases}
$$

If $x \equiv 3, 7 \pmod 8$, then for any solution (k, n, x, y) of (17), we obtain $v_2(T_k(x)) = 0$.

5.5. Formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$

Proof.

The proof is based on some properties of congruences and Lemma 28. □

5.5 Formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$

Lemma 30 (Bérczes, Pink, Savaș and Soydan, 2018)

Assume that k is not even if $x \equiv 5 \pmod 9$. Then we have

$$v_3(T_k(x)) = \begin{cases} v_3(x), & \text{if } k=1, \\ v_3(x)-1, & \text{if } x \equiv 0 \pmod 3 \text{ and } k \geq 2 \text{ is even}, \\ v_3(kx^2), & \text{if } x \equiv 0 \pmod 3 \text{ and } k > 3 \text{ is odd}, \\ v_3(x^2(5x+3)), & \text{if } x \equiv 0 \pmod 3 \text{ and } k = 3, \\ 0, & \text{if } x \equiv \pm 1 \pmod 3 \text{ and } k \geq 3 \text{ is odd}, \\ 0, & \text{if } x \equiv 2,8 \pmod 9 \text{ and } k \geq 2 \text{ is even}, \\ v_3(2x+1)-1, & \text{if } x \equiv 1 \pmod 3 \text{ and } k \geq 2 \text{ is even}. \end{cases}$$

Proof.

On the proof, the main tools are Lemma 10 (Hajdu-2015) and Lemmas 24-25 (Sondow-Tsukerman-2014). □

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5.6 The sketches for the proofs

Now we are ready to prove to the main results.

The proof of Theorem 19.

The main tools are Lemmas 28-29-30 (Formulas for $V_2(T_k(x))$ and $V_3(T_k(x))$) □

The proof of Theorem 20.

The proof is based on Theorem 19. □

The proof of Theorem 21.

In the case $x \in \{2, 3, 6, 7, 10, 11\}$, by using Lemma 27 ; in the case $x \in \{4, 5, 8, 9, 12, 13\}$, by using Theorem 20, it was proved that the eq. (7) has no solution. All computations are supported by MAGMA. □

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- Now, we consider the equation

$$(x-1)^k + x^k + (x+1)^k = y^n \quad x, y \in \mathbb{Z}, \quad n \geq 2. \tag{21}$$

In 2014, it was solved completely by Zhang for $k = 2, 3, 4$ (Actually, firstly, J. W. S.Cassels considered this equation in 1985, and he proved that $x = 0, 1, 2, 24$ are only integer solutions to this equation for $k = 3$ and $n = 2$).

6.1 Some earlier results and motivation

- In 2016, Bennett, Patel and Siksek extended Zhang's result, completely solving equation

$$(x-1)^k + x^k + (x+1)^k = y^n \quad x, y \in \mathbb{Z}, \quad n \geq 2,$$

in the cases $k = 5$ and $k = 6$. The same year, Bennett, Patel and Siksek considered this equation. They gave its integral solutions using linear forms in logarithms, sieving and Frey curves where $k = 3$, $2 \leq r \leq 50$, $x \geq 1$ and n is prime.

- Now we consider a more general. Let d be fixed positive integer. In 2017-2019, Zhang, Koutsianas and Patel studied the integer solutions of the following equation

$$(x-d)^k + x^k + (x+d)^k = y^n, \quad x, y \in \mathbb{Z}, \quad n \geq 2 \qquad (22)$$

for the cases $k = 4$ and $k = 2$, respectively.

- Now we consider a more general. Let d be fixed positive integer. In 2017-2019, Zhang, Koutsianas and Patel studied the integer solutions of the following equation

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for the cases $k = 4$ and $k = 2$, respectively.

- Zhang gave some results on the equation (22) with $k = 4$ by using modular approach. Koutsianas and Patel gave all non-trivial primitive solutions to equation (22) where $k = 2$, n is prime and $d \leq 10^4$.

- Now we consider a more general. Let d be fixed positive integer. In 2017-2019, Zhang, Koutsianas and Patel studied the integer solutions of the following equation

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  for the cases $k = 4$ and $k = 2$, respectively.
- Zhang gave some results on the equation (22) with $k = 4$ by using modular approach. Koutsianas and Patel gave all non-trivial primitive solutions to equation (22) where $k = 2$, n is prime and $d \leq 10^4$.
- Then Garcia and Patel showed that the only solutions to the equation (22) with $n \geq 5$ a prime, $k = 3$, $\gcd(x, d) = 1$ and $0 < d \leq 10^6$ are the trivial ones satifying $xy = 0$.

- Recently, Koutsianas studied the equation (22) with $k = 2$ and $n > 2$ for an infinitely family of d which is an extension of the paper of Koutsianas and Patel.

- Recently, Koutsianas studied the equation (22) with $k = 2$ and $n > 2$ for an infinitely family of d which is an extension of the paper of Koutsianas and Patel.
- He showed that if n is an odd prime, d satisfies

$$d = p^r, \ p \text{ is an odd prime } r \in \mathbb{N}, \tag{23}$$

and $p \leq 10^4$, then all solutions (x, y) of the equation

$$(x-d)^2 + x^2 + (x+d)^2 = y^n, \quad x, y \in \mathbb{N}, \quad \gcd(x, y) = 1 \tag{24}$$

are given in the following table.

6.1 Some earlier results and motivation

Table: Non-trivial primitive solutions (x, y, r, n).

p	(x, y, r, n)
2	$(21, 11, 1, 3)$
7	$(3, 5, 1, 3)$
79	$(63, 29, 1, 3)$
223	$(345, 77, 1, 3)$
439	$(987, 149, 1, 3)$
727	$(2133, 245, 1, 3)$
1087	$(3927, 365, 1, 3)$
3109	$(627, 29, 1, 5)$
3967	$(27657, 1325, 1, 3)$
4759	$(36363, 1589, 1, 3)$
5623	$(46725, 1877, 1, 3)$
8647	$(89187, 2885, 1, 3)$

6. New results on the power values of the sum of three squares in arithmetic progression

- Here we extend the recent results for the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$ $(*)$. We prove the following results:

Theorem 13 (Le and Soydan, 2022)

Let n be an odd prime, and let d be a prime power such that $d = p^r$ $(r \in \mathbb{N})$ and p is an odd prime. If (x, y) is a solution of $(*)$, then $p > 3$ and

$$d = \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i} (-2)^i \right|, \; X_1 \in \mathbb{N}. \tag{25}$$

Moreover, if (25) holds, then the solution (x, y) can be expressed as

$$x = X_1 \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i} (3X_1^2)^{(n-1)/2-i} (-2)^i \right|, \; y = 3X_1^2 + 2. \tag{26}$$

6. New results on the power values of the sum of three squares in arithmetic progression

Theorem 14 (Le and Soydan, 2022)

Under assumption of Theorem 13, the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$ *has at most one solution* (x, y).

6. New results on the power values of the sum of three squares in arithmetic progression

Theorem 14 (Le and Soydan, 2022)

Under assumption of Theorem 13, the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$ *has at most one solution* (x, y).

Theorem 15 (Le and Soydan, 2022)

Let n *be an odd prime. If every odd prime divisor* p *of* d *satisfies* $p \not\equiv \pm 1 \pmod{2n}$, *then the equation* $(x-d)^2 + x^2 + (x+d)^2 = y^n$ *has only the solution* $(x, y, d, n) = (21, 11, 2, 3)$.

6. New results on the power values of the sum of three squares in arithmetic progression

Theorem 14 (Le and Soydan, 2022)

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Let n *be an odd prime. If every odd prime divisor* p *of* d *satisfies* $p \not\equiv \pm 1 \pmod{2n}$, *then the equation* $(x-d)^2 + x^2 + (x+d)^2 = y^n$ *has only the solution* $(x, y, d, n) = (21, 11, 2, 3)$.

Theorem 16 (Le and Soydan, 2022)

If $n > 228000$ *and* $d > 8\sqrt{2}$, *then all solutions* (x, y) *of the equation* $(x-d)^2 + x^2 + (x+d)^2 = y^n$ *satisfy* $y^n < 2^{3/2} d^3$.

Definition 17

For fixed integers a, b, c the homogeneous quadratic polynomial $F = F(x, y) = ax^2 + bxy + cy^2$ is called a binary quadratic form, or simply a form, and is denoted by $\{a, b, c\}$. The integer $d = b^2 - 4ac$ is called the discriminant of the form

Let D_1, D_2, k be fixed positive integers such that $\min\{D_1, D_2\} > 1$, $2 \nmid k$ and $\gcd(D_1, D_2) = \gcd(D_1D_2, k) = 1$, and let $h(-4D_1D_2)$ denote the class number of positive binary quadratic primitive forms with discriminant $-4D_1D_2$.

Lemma 18

If the equation

$$D_1X^2 + D_2Y^2 = k^Z,\ X, Y, Z \in \mathbb{Z},\ \gcd(X, Y) = 1,\ Z > 0$$

has solutions (X, Y, Z), then its every solution (X, Y, Z) can be expressed as

$$Z = Z_1 t,\ t \in \mathbb{N},\ 2 \nmid t,$$
$$X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1(X_1\sqrt{D_1} + \lambda_2 Y_1\sqrt{-D_2})^t,\ \lambda_1, \lambda_2 \in \{1, -1\}.$$

where X_1, Y_1, Z_1 are positive integers such that

$$D_1X_1^2 + D_2Y_1^2 = k^{Z_1},\ \gcd(X_1, Y_1) = 1$$

and $h(-4D_1D_2) \equiv 0 \pmod{2Z_1}$.

Proof.

This is special case of Theorems 1 and 3 of [Le, 1995] for $D < 0$ and $D_1 > 1$. $\square$

Lemma 19 (Le and Soydan, 2022)

If the equation $(x-d)^2+x^2+(x+d)^2=y^n$ *when* n *odd prime and* $d=p^r$, $p>3$ *a prime, has solutions* (x,y), *then* $2\nmid n$ *and its every solution* (x,y) *can be expressed as*

$$x\sqrt{3}+d\sqrt{-2}=\lambda_1(X_1\sqrt{3}+\lambda_2 Y_1\sqrt{-2})^n, \quad \lambda_1,\lambda_2\in\{\pm1\}, \tag{27}$$

$$y=3X_1^2+2Y_1^2,\ X_1,\ Y_1\in\mathbb{N},\ \gcd(X_1,Y_1)=1. \tag{28}$$

6.2.1 Lehmer sequences and primitive divisor theorem

- Using Lemma 18, the proof of Lemma 19 can be done.
- Let α, β be algebraic integers. If $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime integers and α/β is not a root of unity, then (α, β) is called a **Lehmer pair**.

6.2.1 Lehmer sequences and primitive divisor theorem

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- Further, let $A=(\alpha+\beta)^2$ and $C=\alpha\beta$. Then we have

$$\alpha=\frac{1}{2}(\sqrt{A}+\lambda\sqrt{B}),\quad \beta=\frac{1}{2}(\sqrt{A}-\lambda\sqrt{B}),\quad \lambda\in\{\pm 1\},$$

where $B=A-4C$. Such (A,B) is called **the parameters of Lehmer pair** (α,β).

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- Further, let $A = (\alpha+\beta)^2$ and $C = \alpha\beta$. Then we have

$$\alpha = \frac{1}{2}(\sqrt{A} + \lambda\sqrt{B}), \quad \beta = \frac{1}{2}(\sqrt{A} - \lambda\sqrt{B}), \quad \lambda \in \{\pm 1\},$$

  where $B = A - 4C$. Such (A, B) is called **the parameters of Lehmer pair** (α, β).
- Two Lehmer pairs (α_1, β_1) and (α_2, β_2) are called **equivalent** if $\alpha_1/\alpha_2 = \beta_1/\beta_2 \in \{\pm 1, \pm\sqrt{-1}\}$. Obviously, if (α_1, β_1) and (α_2, β_2) are equivalent Lehmer pairs with parameters (A_1, B_1) and (A_2, B_2) respectively, then $(A_2, B_2) = (\varepsilon A_1, \varepsilon B_1)$, where $\varepsilon \in \{\pm 1\}$.

6.2.1 Lehmer sequences and primitive divisor theorem

Definition 20 (Lehmer number)

For a fixed Lehmer pair (α, β), one defines the corresponding sequence of Lehmer numbers by

$$L_m(\alpha, \beta) = \begin{cases} \dfrac{\alpha^m - \beta^m}{\alpha - \beta}, & \text{if } 2 \nmid m, \\ \dfrac{\alpha^m - \beta^m}{\alpha^2 - \beta^2}, & \text{if } 2 | m, \; m \in \mathbb{N}. \end{cases} \tag{29}$$

Then, Lehmer numbers $L_m(\alpha, \beta)$ $(m = 1, 2, ...)$ are nonzero integers. Further, for equivalent Lehmer pairs (α_1, β_1) and (α_2, β_2), we have $L_m(\alpha_1, \beta_1) = \pm L_m(\alpha_2, \beta_2)$ for any m.

6.2.1 Lehmer sequences and primitive divisor theorem

Theorem 21 (Primitive divisor theorem)

Let (α, β) be a Lehmer pair. A prime number q is called a ***primitive divisor*** *of the Lehmer number $L_m(\alpha, \beta)$ if q divides L_m but does not divide $(\alpha-\beta)^2 L_1 \cdots L_{m-1}$. We say that a Lucas sequence is an* ***m-defective Lehmer sequence*** *if L_m* ***has no primitive divisor****.*

Lemma 22 (Voutier, 1995)

Let m be such that $6 < m \leq 30$ and $m \neq 8, 10, 12$. Then up to equivalence, all parameters (A, B) $(A > 0)$ of m-defective Lehmer pairs are given as follows:
(*i*) $m = 7$, $(A, B) = (1, -7), (1, -19), (3, -5), (5, -7), (13, -3), (14, -22)$.
(*ii*) $m = 9$, $(A, B) = (5, -3), (7, -1), (7, -5)$.
(*iii*) $m = 13$, $(A, B) = (1, -7)$.
(*iv*) $m = 14$,
$(A, B) = (3, -13), (5, -3), (7, -1), (7, -5), (19, -1), (22, -14)$.
(*v*) $m = 15$, $(A, B) = (7, -1), (10, -2)$.
(*vi*) $m = 18$, $(A, B) = (1, -7), (3, -5), (5, -7)$.
(*vii*) $m = 24$, $(A, B) = (3, -5), (5, -3)$.
(*viii*) $m = 26$, $(A, B) = (7, -1)$.
(*ix*) $m = 30$, $(A, B) = (1, -7), (2, -10)$.

Lemma 23 (Bilu, Hanrot, Voutier, 2001)

Every positive integer m with $m > 30$ is totally non-defective.

- We now assume that (x, y) is a solution of the equation

$$(x-d)^2 + x^2 + (x+d)^2 = y^n.$$

Then, x, y and d satisfy the equation $3x^2 + 2d^2 = y^n$. From here $p = 3$, we get $3 \mid y$ with $d = p^r$, which contradicts the condition $2 \nmid x$, $2 \nmid y$, $3 \nmid y$, $\gcd(x, d) = 1$. So we have $p > 3$.

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- By Lemma 19, we have

$$x = X_1 \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i} (3X_1^2)^{(n-1)/2-i} (-2Y_1^2)^i \right| \tag{30}$$

and

$$d = Y_1 \left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i} (-2Y_1^2)^i \right|. \tag{31}$$

- Since $d = p^r$, by (31), we get

$$Y_1 = p^s,\ s \in \mathbb{Z},\ 0 \leq s \leq r \tag{32}$$

and

$$\left| \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i} (-2Y_1^2)^i \right| = p^{r-s}. \tag{33}$$

Let

$$\alpha = X_1\sqrt{3} + Y_1\sqrt{-2},\ \beta = X_1\sqrt{3} - Y_1\sqrt{-2}. \tag{34}$$

- Since $d = p^r$, by (31), we get

$$Y_1 = p^s,\ s \in \mathbb{Z},\ 0 \leq s \leq r \tag{32}$$

and

$$\left|\sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i}(-2Y_1^2)^i\right| = p^{r-s}. \tag{33}$$

Let

$$\alpha = X_1\sqrt{3} + Y_1\sqrt{-2},\ \beta = X_1\sqrt{3} - Y_1\sqrt{-2}. \tag{34}$$

- So, we have

$$\alpha + \beta = 2X_1\sqrt{3},\ \alpha - \beta = 2Y_1\sqrt{-2},\ \alpha\beta = y. \tag{35}$$

Hence, we see (checking necessary conditions) such that (α, β) is a Lehmer pair with the parameters

$$(A, B) = (12X_1^2, -8Y_1^2). \tag{36}$$

- Further, let $L_m(\alpha, \beta)$ $(m = 1, 2, \cdots)$ be the corresponding Lehmer numbers. By the definition of Lehmer number, we have

$$\sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (3X_1^2)^{(n-1)/2-i} (-2Y_1^2)^i = L_n(\alpha, \beta). \tag{37}$$

Therefore, we get

$$|L_n(\alpha, \beta)| = p^{r-s}. \tag{38}$$

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Therefore, we get

$$|L_n(\alpha, \beta)| = p^{r-s}. \tag{38}$$

- If $s > 0$, by primitive divisor theorem, then the Lehmer number $L_n(\alpha, \beta)$ has no primitive divisors. Therefore, since n is an odd prime, by Lemma 22 [Voutier 1999] and Lemma 23 [Bilu, Hanrot, Voutier, 2001], we find from $(A, B) = (12X_1^2, -8Y_1^2)$ that $n \in \{3, 5\}$. When $n = 3$, by (32) and (33), we have

$$9X_1^2 - 2p^{2s} = \pm p^{r-s}. \tag{39}$$

- When $n = 5$, by (32) and (33), we have

$$45X_1^4 - 60X_1^2 p^{2s} + 4p^{4s} = \pm p^{r-s}. \tag{40}$$

In both cases, using elementary arguments (via congruences and Legendre symbol), we get contradictions. Then, we get $s = 0$ which means that $Y_1 = 1$. Thus, the the proof of Theorem 13 is completed.

- Under the assumption of Theorem 13, by elementary arguments, we prove Theorem 14.

Lemma 24 (Lehmer, 1930)

If n is an odd prime and q is a prime divisor of the Lehmer number $L_n(\alpha, \beta)$, then $q \equiv \pm 1 \pmod{2n}$.

- By Lemma 19, if (x, y) is a solution of the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$, then by following similar steps of proof of Theorem 13, we have

$$d = Y_1 |L_n(\alpha, \beta)|. \tag{41}$$

Lemma 24 (Lehmer, 1930)

If n is an odd prime and q is a prime divisor of the Lehmer number $L_n(\alpha, \beta)$, then $q \equiv \pm 1 \pmod{2n}$.

- By Lemma 19, if (x, y) is a solution of the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$, then by following similar steps of proof of Theorem 13, we have

$$d = Y_1 |L_n(\alpha, \beta)|. \tag{41}$$

- Since n is an odd prime and every odd prime divisor p of d satisfies $q \not\equiv \pm 1 \pmod{n}$, by Lemma 24, we get from (41) that

$$|L_n(\alpha, \beta)| = 1 \tag{42}$$

and

$$Y_1 = d. \tag{43}$$

- From here, we see that the Lehmer number $L_n(\alpha, \beta)$ has no primitive divisors. Therefore, using the same method as in the proof of Theorem 13, by Lemma 22 [Voutier, 1999] and Lemma 23 [Bilu, Hanrot, Voutier, 1999], we can deduce from $|L_n(\alpha, \beta)| = 1$ and $Y_1 = d$ that $n \in \{3, 5\}$.

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- In both cases, using elementary arguments (via congruences), we get contradictions. So, the proof of theorem is completed.

- Now we are interested in obtaining a lower bound for n on the equation $(x-d)^2+x^2+(x+d)^2=y^n$, so we need Baker's theory.

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Definition 25 (Absolute logarithmic height)

Let θ be any non-zero algebraic number with minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{\ell}(X-\theta^{(j)})$ which is of degree ℓ over $\mathbb{Q}$. We denote by

$$h(\theta)=\frac{1}{\ell}\left(\log|a|+\sum_{j=1}^{\ell}\log\max\{1,|\theta^{(i)}|\}\right)$$

its absolute logarithmic height where $(\theta^{(j)})_{1\le j\le\ell}$ are conjugates of θ.

Lemma 26 (Appendix of Bilu, Hanrot, Voutier 2001)

Let θ be a complex algebraic number with $|\theta| = 1$, and θ is not root of unity. Let b_1, b_2 be positive integers, and let $\Lambda = b_1 \log\theta - b_2\pi\sqrt{-1}$. Then we have

$$\log|\Lambda| > -(9.03H^2 + 0.23)(Dh(\theta) + 25.84) - 2H - 2\log H - 0.7D + 2.07,$$

where $D = [\mathbb{Q}(\theta) : \mathbb{Q}]/2$, $H = D(\log B - 0.96) + 4.49$, $B = \max\{13, b_1, b_2\}$.

- By Lemma 19, if (x, y) is a solution of the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$, then

$$d = \frac{1}{2\sqrt{2}}|\alpha^n - \beta^n|, \tag{44}$$

where α, β are defined as in (34). By $y = 3X_1^2 + 2Y_1^2$ and the choice of α and β, we have

$$|\alpha| = |\beta| = \sqrt{y}. \tag{45}$$

- By Lemma 19, if (x, y) is a solution of the equation $(x-d)^2 + x^2 + (x+d)^2 = y^n$, then

$$d = \frac{1}{2\sqrt{2}}|\alpha^n - \beta^n|, \qquad (44)$$

where α, β are defined as in (34). By $y = 3X_1^2 + 2Y_1^2$ and the choice of α and β, we have

$$|\alpha| = |\beta| = \sqrt{y}. \qquad (45)$$

- Let $\theta = \alpha/\beta$. θ is a complex algebraic number with $|\theta| = 1$, θ is not a root of unity and

$$h(\theta) = \frac{1}{2}\log y. \qquad (46)$$

- By $d = \frac{1}{2\sqrt{2}}|\alpha^n - \beta^n|$ and $|\alpha| = |\beta| = \sqrt{y}$, we have

$$d = \frac{1}{2\sqrt{2}}|\beta^n| \left| \left(\frac{\alpha}{\beta}\right)^n - 1 \right| = \frac{1}{2\sqrt{2}} y^{n/2} |\theta^n - 1|. \quad (47)$$

For any complex number z, we have either $|e^z - 1| \geq \frac{1}{2}$ or $|e^z - 1| \geq \frac{2}{\pi}|z - t\pi\sqrt{-1}|$ for some integers t.

- By $d = \frac{1}{2\sqrt{2}}|\alpha^n - \beta^n|$ and $|\alpha| = |\beta| = \sqrt{y}$, we have

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  For any complex number z, we have either $|e^z - 1| \geq \frac{1}{2}$ or $|e^z - 1| \geq \frac{2}{\pi}|z - t\pi\sqrt{-1}|$ for some integers t.

- Put $z = n\log\theta$. We get either

$$|\theta^n - 1| \geq \frac{1}{2} \tag{48}$$

  or

$$|\theta^n - 1| \geq \frac{2}{\pi}|n\log\theta - t\pi\sqrt{-1}|, \ t \in \mathbb{N}, \ t \leq n. \tag{49}$$

  If (48) holds, since $d > 8\sqrt{2}$, then from (47) we obtain $y^n \leq 32d^2 < 2^{3/2}d^3$ and the theorem is true.

- Let

$$\Lambda = n\log\theta - t\pi\sqrt{-1}. \tag{50}$$

By some inequalities, we have

$$d \geq \frac{y^{n/2}}{\pi\sqrt{2}}|\Lambda|. \tag{51}$$

- Let
$$\Lambda = n \log \theta - t\pi\sqrt{-1}. \tag{50}$$
By some inequalities, we have
$$d \geq \frac{y^{n/2}}{\pi\sqrt{2}}|\Lambda|. \tag{51}$$

- If $y^n \geq 2^{3/2}d^3$, then from (51) we get
$$\pi \geq y^{n/6}|\Lambda|,$$
whence we obtain
$$\log \pi \geq \frac{n}{6}\log y + \log|\Lambda|. \tag{52}$$
Notice that $[\mathbb{Q}(\theta) : \mathbb{Q}] = 2$, $n \geq t$ and $n > 228000$.

- Applying Lemma 26 [Appendix of Bilu, Hanrot, Voutier 2001] to $h(\theta) = \frac{1}{2}\log y$, by $\Lambda = n\log\theta - t\pi\sqrt{-1}$, we have

$$\log|\Lambda| > -(9.03H^2+0.23)(\frac{1}{2}\log y+25.84)-2H-2\log H+1.37, \quad (53)$$

where

$$H = \log n + 3.53. \quad (54)$$

- Applying Lemma 26 [Appendix of Bilu, Hanrot, Voutier 2001] to $h(\theta) = \frac{1}{2}\log y$, by $\Lambda = n\log\theta - t\pi\sqrt{-1}$, we have

$$\log|\Lambda| > -(9.03H^2+0.23)(\frac{1}{2}\log y+25.84)-2H-2\log H+1.37, \quad (53)$$

where

$$H = \log n + 3.53. \quad (54)$$

- Combining the above inequality with some inequalities, we get $n < 228000$, a contradiction. Thus, if $n > 228000$ and $d > 8\sqrt{2}$, then $y^n < 2^{3/2}d^3$. The theorem is proved.

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7. References

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Thank you for your attention!

Merci pour votre attention!